A NEW UNCOUNTABLY CATEGORICAL GROUP

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ABSTRACT. We construct an uncountably categorical group with a geometry
that is not locally modular. It is not possible to interpret a field in this group.
We show the group is CM-trivial.

1. INTRODUCTION

Let T be a countable complete first order theory. T is called λ-*categorical* if
all models of T of cardinality λ are isomorphic. In [15] M. D. Morley proved J.
Łoś's conjecture that a theory T that is categorical in some uncountable power is
categorical in every uncountable power. These *uncountably categorical* theories are
the simplest case of a theory T that allows the classification of its models in the
sense of S. Shelah [22]. The main examples of such theories are the theories of
algebraically closed fields of a given characteristic. The classification of the models
of an uncountably categorical theory is given by the unique size of a "transcendence
basis" of the models, similar to the classification of algebraically closed fields by
Steinitz [1].

For any uncountably categorical theory T one can find a formula $\phi(x)$ (with
auxiliary parameters) which is *strongly minimal* in the sense that the set defined by
ϕ in any model of T cannot be split into two infinite definable sets. The appropriate
generalization of the notion of "transcendence degree" involves a combinatorial
geometry with the Steinitz exchange property, which is defined on any strongly
minimal set.

Let D be any set. Assume that cl is an operator from the set of finite subsets
of D into the set of subsets of D. We say that cl defines a *pregeometry* on D if
the following conditions are fulfilled. Let a, b be elements of D and A, B be finite
subsets of D:

  i) $A \subseteq cl(A)$.
  ii) $A \subseteq B$ implies $cl(A) \subseteq cl(B)$.
  iii) $cl(cl(A)) = cl(A)$.
  iv) If $a \in cl(A \cup \{b\}) \setminus cl(A)$, then $b \in cl(A \cup \{a\})$.

A pregeometry is called a *geometry* if $cl(\{a\}) = \{a\}$ for each point a. One may
pass from a pregeometry to a geometry by removing $cl(\emptyset)$ and taking the closures
of the remaining points as the points of a geometry. There is a natural geometry
on any strongly minimal set in which $cl(A)$ is the union of all finite A-definable
sets; this is called the *algebraic closure* in the sense of model theory. Hence every

strongly minimal formula of T defines a geometry for T, and all the geometries obtained in this way are isomorphic up to finite localizations. We speak of the geometry of T.

Let us mention the following examples:

1) The theory of the successor function on the naturals. The corresponding geometry is *disintegrated*, i.e.: $a \in cl(A)$ if and only if $a \in A$.

2) The theory of a vector space where the language contains $+$, $-$, 0 and a unary function for each element of the base field, corresponding to scalar multiplication by that element. The corresponding geometry is the projective geometry. This geometry is *modular*: If X and Y are closed sets, then the geometrical dimension d fulfills the following equation:

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y).$$

3) The theory of an algebraically closed field. The corresponding geometry is not *locally modular*. That means even a localization of the geometry at a finite set is not modular.

First B. Zil'ber [25] tried to classify uncountably categorical theories T according to their geometries. He proved:

- If the geometry is disintegrated, then no group is interpretable in T.

Then E. Hrushovski showed:

- If the geometry is not disintegrated but still locally modular, then a group but no field is interpretable in T.

B. Zil'ber conjectured that in the the remaining case (the geometry is not locally modular) a field is interpretable in T. In 1988 E. Hrushovski [12] found a counterexample of an uncountably categorical theory with a non-locally-modular geometry and without the possibility of interpreting a group. The aim of this paper is to construct a pure group with an uncountably categorical theory and a non-locally-modular geometry that does not allow the interpretation of a field. "Pure group" means that we work in the language of group theory only.

There are only two possibilities in searching for such a group G. Either G is a nilpotent group of bounded exponent, or a quotient of definable subgroups of G is a counterexample to the Cherlin–Zil'ber Conjecture that every simple group of finite Morley rank is an algebraic group over an algebraically closed field. This is shown in section 2. We consider nilpotent groups of class 2 with exponent p where p is a prime greater than 2. Note that an uncountably categorical pure group has a locally modular geometry if and only if it is abelian by finite [13].

We transfer the construction of a Hrushovski-geometry in relational languages [12] (see also [10]) to the case of a pure group. The most difficult part is the proof of the corresponding Amalgamation Theorems.

The final group is obtained by the method of successive amalgamation as introduced by R. Fraïssé [9]. We want to amalgamate finite nilpotent groups of class 2 of exponent $p > 2$ with the additional property that the commutator subgroup is the center. But it is more convenient to work in an auxiliary category $\mathcal{S}$ whose objects are vector spaces V over a finite field F_p of order p together with a distinguished subspace $N(V)$ of the exterior square $\wedge^2 V$ (see section 3). There is a functorial correspondence between the groups above and $\mathcal{S}$ that is 1-1 at the level of isomorphism types of objects. We lose some morphisms.

In the relational case E. Hrushovski defined a function

$$\delta(A) = \text{ number of elements} - \text{number of relations}$$

for all finite substructures A of the structure under consideration. The corresponding function in our construction is

$$\delta(H) = dim(H) - dim(N(H))$$

where H is a finite subspace of the underlying vectorspace V of a structure in $\mathcal{S}$ and dim is the linear dimension. Note that by definition $N(H) = N(V) \cap \wedge^2 H$. We only consider structures in $\mathcal{S}$ that fulfill the following condition:

($\Sigma 3$): For all finite subspaces H of V we have $\delta(H) \geq 3$ if $dim(H) \geq 3$, and $\delta(H) = dim(H)$ otherwise.

As in the relational case, δ can be used to define a pregeometry cl on V provided ($\Sigma 3$) is fulfilled. We give a short description of the definition of cl. A finite subspace H of V is called *selfsufficient* if $\delta(K) \geq \delta(H)$ for all finite K with $H \subseteq K \subseteq V$. Every finite subspace H is contained in the smallest finite selfsufficient subspace $CSS(H)$ that is contained in every selfsufficient extension of H. Then we can define the desired pregeometry by

$$a \in cl(a_1, \dots, a_n) \quad \text{iff} \quad \delta(CSS(\langle\{a_1, \dots, a_n\}\rangle)) = \delta(CSS(\langle\{a, a_1, \dots, a_n\}\rangle)).$$

The details of the definitions of all notions above are given in section 3.

Now assume that A is a subspace of B and both are finite subspaces of V in some $\mathcal{S}$-structure. Then B is called a *minimal* extension of A if $\delta(A) = \delta(B)$ and $\delta(K) > \delta(A)$ for all K with $A \subseteq K \subseteq B$ and $A \neq K \neq B$. To describe the amalgamation process let A, B, and C be finite structures in $\mathcal{S}$ such that A is a common substructure of B and C, and B is a minimal extension of A. If A is selfsufficient in C and B is not realized in C over A, then we replace C by the following free amalgam $D = B \star_A C$ of B and C over A: The underlying vectorspace D is $B \oplus_A C$, and $N(D)$ is the subspace $N(B) + N(C)$ of $\wedge^2 D$. The First Amalgamation Theorem in section 4 ensures that D satisfies ($\Sigma 3$) again, provided that A, B, and C fulfil ($\Sigma 3$). Section 5 contains some preparations for the argument in section 6.

Let $(M, N(M))$ be the final $\mathcal{S}$-structure produced by the amalgamation process described above. Let G be the corresponding group and let $\overline{G}$ be $G/Z(G)$ where $Z(G)$ is the center of G. Then we can identify the vector spaces M and $\overline{G}$. We have to modify the construction in such a way that we get $cl = acl$ on $\overline{G}$, where cl is the pregeometry given by $(M, N(M))$ and acl is the algebraic closure on $\overline{G}$ given by the group G. For this it is sufficient that the number of realizations of a minimal extension K over H has a bound which depends only on $dim(K)$. The complicated condition ($\Sigma 4$) implies such a bound. In section 6 we prove that ($\Sigma 4$) is preserved by the described amalgamation (Second Amalgamation Theorem). Then we can show that the theory Σ of the corresponding group will be uncountably categorical and its geometry will be given by cl on M. In section 7 the construction of the group and the final results are given apart from the non-interpretability of a field. This will be done in section 8. We show CM-triviality of the theory Σ. CM-triviality was introduced by E. Hrushovski in [12]. This implies the desired non-interpretability of a field. In the first version of this paper we proved directly that every group H that we can interpret in our group is nilpotent by finite and bounded.

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2. On groups of finite Morley rank

If G is a group, then let $[X, Y]$ be the set $\{[a, b] : a \in X, b \in Y\}$. For subsets $X \subseteq G$ let $\langle X \rangle$ be the subgroup generated by X. Hence we can define the commutator subgroup G' as $\langle [X, Y] \rangle$. We use $Z(G)$ to denote the center of G.

Uncountably categorical theories are ω-stable of finite Morley rank. A group is called *connected* if it contains no proper definable subgroup of finite index. An ω-stable group G contains a definable connected normal subgroup G^0 of finite index. It is called the *connected component* of G.

There is a well-known conjecture of G. Cherlin and B. Zil'ber that every simple infinite group of finite Morley rank is an algebraic group over an algebraically closed field. Combining a number of known results, we obtain the following:

Theorem 2.1. *Let G be a group of finite Morley rank that does not allow the interpretation of an infinite field. Either we get a counterexample to the Cherlin - Zil'ber - Conjecture as a quotient of definable subgroups of G, or $G^0 = A * B$ where A is Abelian and B is a nilpotent definable subgroup of bounded exponent. (Here $*$ denotes a central product.)*

Proof. Even in the superstable case there are definable subgroups $H_0 \lhd H_1 \lhd \ldots \lhd H_r = G^0$ of G where H_{i+1}/H_i is infinite and furthermore either abelian or simple over a finite center. In the case of finite Morley rank this follows from Zil'ber's Indecomposability Theorem ([23], see [7]). The proof for superstable groups is found in [5]. Hence we either have an infinite simple group as a quotient of definable subgroups of G, or else G^0 is soluble. We need the following result of Zil'ber [24]:

Theorem 2.2. *In every connected soluble non-nilpotent group of finite Morley rank it is possible to interpret a field.*

In a simple algebraic group over an algebraically closed field one can find a definable connected soluble non-nilpotent subgroup. By Theorem 2.2 a field is definable. Hence if we get a simple group above, then it is a counterexample to the Cherlin-Zil'ber Conjecture. Otherwise we know that G^0 is soluble. By Theorem 2.2 again we obtain that G^0 is nilpotent.

Note that B. Zil'ber [26] has shown that every torsion-free nilpotent uncountably categorical group is an algebraic group over an algebraically closed field. C. Grünenwald and F. Haug obtained the following (Theorem 2.3 in [11]):

Theorem 2.3. *Let G be a non-Abelian nilpotent stable group. Assume that the commutator subgroup G' is torsion-free. Then an infinite field of characteristic 0 is interpretable.*

Furthermore we need the following result of A. Nesin [16]:

Theorem 2.4. *Let G be an ω-stable nilpotent group. Then there are definable characteristic subgroups D and B such that D is divisible, B has finite exponent, $G = DB$, and $[D, B] = \langle 1 \rangle$. D can be decomposed as $D = D^+ \oplus F$, where D^+ is the central subgroup of torsion elements of D and F is a torsion-free subgroup.*

Now our group G^0 has the structure described by Theorem 2.4. If D is Abelian, then D is central and the assertion follows. Otherwise consider the definable subgroup D. Then $D' = F'$. F' is torsion-free and non-trivial. By Theorem 2.3 it is possible to interpret a field, a contradiction. $\square$

Theorem 2.1 gives the motivation to consider only groups G, that satisfy the following property:

($\Sigma1$): G is nilpotent of class 2 and of exponent p, where p is a fixed prime greater than 2.

Let $F_c(p,\kappa)$ be the free group in the variety of nilpotent groups of class c and of exponent p ($p > c$). κ is the number of free generators. Let κ be infinite. In [2] it is shown that $F_c(p,\kappa)$ has an ω-stable theory with c dimensions. Thus $F_2(p,\kappa)$ is not so far from the group we desire; but this group has infinite Morley-rank. The reason for its two dimensions and the infinite Morley-rank is that there is no integer n such that every element of the commutator subgroup G' can be presented as a product of at most n commutators. In a group G of finite Morley-rank we get, as a consequence of B. Zil'ber's Indecomposability Theorem, that there are $g_1, \ldots, g_k \in G$ such that

$$G' = [g_1, G][g_2, G] \ldots [g_k, G].$$

Our final group will satisfy this condition in a very strong form:

($\Sigma2$): $\forall z \in Z(G)\, \forall x \notin Z(G)\, \exists y\, (z = [x,y])$.

All the groups G considered here will satisfy ($\Sigma1$) and a weak form of ($\Sigma2$):

($\Sigma2$)^w: $Z(G) = G'$.

Note however that this condition is not elementary.

3. THE GEOMETRY

As above all the groups G considered here will fulfill ($\Sigma1$) and ($\Sigma2$)^w. Let $\overline{G}$ be $G/Z(G)$. If G satisfies ($\Sigma1$) and ($\Sigma2$)^w, then $[x,y]$ defines an alternating bilinear map of $\overline{G} \times \overline{G}$ into $Z(G)$ such that $[\overline{G}, \overline{G}]$ generates $Z(G)$. We will show that G is determined up to isomorphism by this bilinear map, using the fact that $\overline{G}$ and $Z(G)$ are vector spaces over the field F_p with p elements.

Lemma 3.1. *Let G and H be groups with the properties ($\Sigma1$) and ($\Sigma2$)^w. Let $\overline{G} = G/Z(G)$, $\overline{H} = H/Z(H)$, and let $[\ ,\ ]_G$, $[\ ,\ ]_H$ be the corresponding commutator maps. Assume there is an isomorphism $f^B : (\overline{G}, Z(G); [\ ,\ ]_G) \simeq (\overline{H}, Z(H); [\ ,\ ]_H)$. Let $\{a_0, a_1, \ldots\}$ be a basis of the vector space $\overline{G}$ and choose $c_i \in G$ in the coset modulo $Z(G)$ corresponding to a_i. Let $f(c_i)$ be any element in H in the coset modulo $Z(H)$ corresponding to $f^B(a_i)$.*
Then f^B extends to an isomorphism f of G onto H.

Proof. G can be reconstructed from $(\overline{G}, Z(G); [\ ,\ ]_G)$ in the following way: Let G^* be the following group. The elements are tuples $(\sum_\alpha r_\alpha a_\alpha, z)$, where $z \in Z(G)$ and $r_\alpha = 0$ up to finitely many α. The group multiplication is defined by

$$\left(\sum_\alpha r_\alpha a_\alpha, z_1\right) \circ \left(\sum_\alpha s_\alpha a_\alpha, z_2\right) = \left(\sum_\alpha (r_\alpha + s_\alpha) a_\alpha, z_1 + z_2 + \sum_{\beta < \alpha} r_\alpha s_\beta [a_\alpha, a_\beta]\right).$$

Then G and G^* are isomorphic. To prove this we write elements of G uniquely as $\prod_\alpha c_\alpha^{r_\alpha} z$, where $z \in Z(G)$ and $r_\alpha = 0$ up to finitely many α. $g(\prod_\alpha c_\alpha^{r_\alpha} z) =$

$(\sum_\alpha r_\alpha g_\alpha, z)$ gives the desired isomorphism of G and G^* with $g(c_i) = (a_i, 0)$. Analogously we obtain an isomorphism h of H onto H^* with $h(f(c_i)) = (f^B(a_i), 0)$. If f^+ is the isomorphism of G^* onto H^* given by f^B, then $f = h^{-1}f^+g$ has the desired properties. □

Let $\mathcal{G}$ be the category of all groups that satisfy $(\Sigma1)$ and $(\Sigma2)^w$ with group monomorphisms as morphisms. Let $\mathcal{B}$ be the category of all alternating bilinear maps $\beta : V \times V \to W$, where V and W are vector spaces over the field F_p, and the image of β generates W (this is an analog of condition $(\Sigma2)^w$). The morphisms from $(V_1, W_1; \beta_1)$ to $(V_2, W_2; \beta_2)$ are pairs (f, g) with $f : V_1 \to V_2$, $g : W_1 \to W_2$ so that $\beta_2 f = g\beta_1$. The correspondence between groups G in $\mathcal{G}$ and the associated structures $(\overline{G}, Z(G); [,]_G)$ in $\mathcal{B}$ rise to a functor from $\mathcal{G}$ onto $\mathcal{B}$; this correspondence is 1-1 at the level of objects, by the proof of Lemma 3.1. (Lemma 3.1 does not say that the map is onto, but the proof shows it.)

For every vector space V there is a "free bilinear map over V", namely $\wedge : V \times V \to \wedge^2 V$ in $\mathcal{B}$, characterized by the following universal property:

$(\wedge)$: For every alternating bilinear map $\beta : V \times V \to W$ there is a unique linear $f_\beta : \wedge^2 V \to W$ such that $f_\beta \wedge = \beta$.

The vector space $\wedge^2 V$ (equipped with the associated bilinear map $\wedge$) is called the *exterior square* of V (cf. [14]). We denote the kernel of f_β on $\wedge^2 V$ by $N(\beta)$, or more commonly, abusing the notation, by $N(V)$. For any structure $(V, W; \beta)$ in $\mathcal{B}$, there is a canonical isomorphism $(V, W; \beta) \simeq (V, \wedge^2 V/N(\beta); \pi)$ where π is the composition of the canonical maps $V \times V \to \wedge^2 V \to \wedge^2 V/N(\beta)$. These considerations lead to the introduction of another category $\mathcal{S}$ equivalent to $\mathcal{B}$. The objects of $\mathcal{S}$ are pairs (V, N) with $N \subseteq \wedge^2 V$ and the maps $f : (V_1, N_1) \to (V_2, N_2)$ such that the induced map $\wedge^2 f$ between the exterior squares satisfies: $\wedge^2 f[N_1] = \wedge^2 f[V_1] \cap N_2$. The main work in this paper is done in $\mathcal{S}$. Let $(M, N(M))$ be an element of $\mathcal{S}$. Often we write M only. For $H \leq M$ we let $N(H) = \wedge^2 H \cap N(M)$, where of course $\wedge^2 H$ is canonically identified with a subspace of $\wedge^2 M$. Note that $(H, N(H)) \in \mathcal{S}$. By the equivalence of the categories $\mathcal{B}$ and $\mathcal{S}$ Lemma 3.1 implies the following:

Corollary 3.2. *Let G and H be groups in $\mathcal{G}$. Assume there is an isomorphism f_0 of a subgroup $G_0 \in \mathcal{G}$ of G onto a subgroup $H_0 \in \mathcal{G}$ of H. Let M and N be $\mathcal{S}$-structures corresponding to G and H respectively. Let M_0 be the substructure of M corresponding to G_0 and let N_0 be the substructure of N corresponding to H_0. Assume there is an $\mathcal{S}$-isomorphism g of M onto N such that the restriction of g to M_0 is the $\mathcal{S}$-isomorphism induced by f_0. Then there is an isomorphism f of G onto H that extends f_0 and induces the given $\mathcal{S}$-isomorphism g of M onto N.*

With M fixed, we let δ be the function defined on finite dimensional subspaces H of M by $\delta(H) = dim(H) - dim(N(H))$. We are interested in groups with relatively few relations. First we formulate this requirement for $\mathcal{S}$:

$(\Sigma3)$: Let $(M, N(M))$ be an element of $\mathcal{S}$. For every finite subspace H of M, $\delta(H) \geq min\{3, dim(H)\}$.

Sometimes a weaker condition is sufficient:

$(\Sigma3)^0$: For every finite subspace H of M, $\delta(H) \geq 0$.

Let G be a group in $\mathcal{G}$. We say that G fulfills $(\Sigma3)$ if the corresponding structure in $\mathcal{S}$ satisfies $(\Sigma3)$. In the language of groups we can formulate this by the following statement. It is possible to express this by a set of elementary sentences.

($\Sigma3$): Let G_0 be any finite subgroup of G. Consider $\overline{G_0}$ and G_0' as vector spaces over F_p. If $dim(\overline{G_0}) = n$, then

$$dim(G_0') \geq min\{(n/2)(n-1) - n + 3, (n/2)(n-1)\}.$$

We see that we can also speak about $\delta(H)$ for subspaces H of $\overline{G}$.

With this definition of δ we can define a geometry on $\mathcal{S}$-structures M following the approach of E. Hrushovski [12] for relational structures. We use the presentation of J.B. Goode and B. Poizat in [10]. Then we have the same geometry on $\overline{G}$ for the corresponding G in $\mathcal{G}$.

Definition. For $(M, N(M))$ in $\mathcal{S}$ and $H \leq M$, H is *selfsufficient* in M if for all finite subspaces K with $H \leq K \leq M$ we have $\delta(H) \leq \delta(K)$.

If $(\Sigma3)^0$ is true for M, then every finite subspace of M is contained in a finite selfsufficient subspace of M.

Lemma 3.3. *For subspaces H and K of M, $N(H \cap K) = N(H) \cap N(K)$.*

Proof. This holds because $\wedge^2 H \cap \wedge^2 K = \wedge^2(H \cap K)$. $\square$

Lemma 3.4. *Let H and K be finite subspaces of M Then*

$$\delta(H + K) \leq \delta(H) + \delta(K) - \delta(H \cap K).$$

Proof. $dim(H+K) = dim(H) + dim(K) - dim(H \cap K)$, and $dim(N(H) + N(K)) = dim(N(H)) + dim(N(K)) - dim(N(H \cap K))$, using Lemma 3.3. As $N(H) + N(K) \leq N(H + K)$, our claim follows from the definition of δ. $\square$

To define a pregeometry on M (on $\overline{G}$ respectively) we only need $(\Sigma3)^0$, Lemma 3.4, and $\delta(\emptyset) = 0$ (see [12]). This definition is described in the following.

Corollary 3.5. *Let H and K be finite subspaces of M.*
  1. *If $\delta(H \cap K) > \delta(H)$, then $\delta(H + K) < \delta(K)$.*
  2. *If $\delta(H \cap K) \geq \delta(H)$, then $\delta(H + K) \leq \delta(K)$.*

Lemma 3.6. *If H and K are selfsufficient subspaces of M, then $A = H \cap K$ is selfsufficient in M.*

Proof. Otherwise consider a counterexample $A = H \cap K \neq \langle 0 \rangle$ such that $\delta(K) \leq \delta(H)$ and $\delta(K)$ is minimal. If $\delta(A) > \delta(K)$ then, by Corollary 3.5, $\delta(K+H) < \delta(H)$. This contradicts the selfsufficiency of H. Therefore we can assume that $\delta(A) \leq \delta(K)$. There exists $B \supseteq A$ such that B is selfsufficient and $\delta(B) < \delta(A) \leq \delta(K)$. Since $\delta(B) < \delta(K)$ and $\delta(K)$ was chosen minimal for a counterexample, it follows that $B \cap K$ is selfsufficient. Note that $B \cap K \supseteq A \neq \langle 0 \rangle$. We obtain a new counterexample:

$$A = (B \cap K) \cap H, \text{ where } B \cap K \text{ and } H \text{ are selfsufficient.}$$

Then $\delta(B \cap K) \leq \delta(B) < \delta(K)$, contradicting the minimality of $\delta(K)$. $\square$

Let us assume for the rest of this section that M satisfies $(\Sigma3)^0$. Hence every finite subspace A of M is contained in a finite selfsufficient subspace B of M. Therefore and because of Lemma 3.6 we can define:

Definition. Let A be a finite subspace of M.
  1. The *selfsufficient closure* $CSS(A)$ of A is the intersection of all finite selfsufficient subspaces B with $A \leq B$.

2. The *geometrical dimension* $d(A)$ of A is $\delta(CSS(A))$.
3. For $a_1, \dots, a_n, a$ in M let us define $a \in cl(\{a_1, \dots, a_n\})$ if and only if

$$d(\langle \{a_1, \dots, a_n, a\} \rangle) = d(\langle \{a_1, \dots, a_n\} \rangle).$$

By the definition, for all finite subspaces A and B of M with $A \subseteq B$ we have $\delta(CSS(A)) \leq \delta(B)$, especially $\delta(CSS(A)) \leq \delta(A)$. We use the same notions in $\overline{G}$ for $G \in \mathcal{G}$. If A is a finite subspace of $\overline{G}$, then in G^{eq} $CSS(A)$ is part of the algebraic closure of A. If $m = dim(CSS(A))$ and $n = \delta(CSS(A))$, then $CSS(A)$ is the subspace X of $\overline{G}$ that contains A, has linear dimension m, and has $\delta(X) = n$. Using this we can write a formula that says "$x \in X$". This formula uses elements of G that represent A modulo $Z(G)$ as parameters.

Lemma 3.7. *Let A and B be subspaces of M with $A \subseteq B$. Then $d(A) \leq d(B)$.*

Proof. $A \subseteq B \subseteq CSS(B)$ implies $CSS(A) \subseteq CSS(B)$. $\qquad\square$

Lemma 3.8. *If $d(A + \langle a \rangle) > d(A)$ then $d(A + \langle a \rangle) = d(A) + 1$.*

Proof. W.l.o.g. $CSS(A) = A$. By definition $d(A + \langle a \rangle) = \delta(CSS(A + \langle a \rangle)) \leq \delta(A + \langle a \rangle)$. It is sufficient to show that $\delta(A + \langle a \rangle) \leq \delta(A) + 1$. This follows from Lemma 3.4: $\delta(A + \langle a \rangle) \leq \delta(A) + \delta(\langle a \rangle) = \delta(A) + 1$. $\qquad\square$

Theorem 3.9. *cl defines a pregeometry on M.*

Proof. (1) $\{a_1, \dots, a_n\} \subseteq cl(a_1, \dots, a_n)$ is true by the definition. (2) For finite subsets $A \subseteq B$ we obtain $cl(A) \subseteq cl(B)$.

Proof of (2). W.l.o.g. A and B are subspaces. Assume that $a \in cl(A)$. Then $d(A + \langle a \rangle) = d(A)$. Choose a selfsufficient K with $A \cup \{a\} \subseteq K$ and $\delta(K) = d(K) = d(A + \langle a \rangle) = d(A)$. Choose $B \subseteq H$ such that H is selfsufficient and $\delta(H) = d(B)$. By Lemma 3.7 $d(A) \leq d(B)$. Therefore $\delta(K) = d(A) \leq d(B) = \delta(H)$. We have $A \subseteq K \cap H$. By Lemma 3.6 $K \cap H$ is selfsufficient. It follows that $d(A) \leq d(K \cap H) \leq d(K) \leq d(A)$. That means $\delta(K \cap H) = \delta(K)$. By Corollary 3.5 $\delta(K + H) \leq \delta(H) = d(B)$. Therefore $d(B) \leq d(K + H) \leq \delta(K + H) \leq d(B)$. Since $B \cup \{a\} \subseteq K + H$, we get $d(B + \langle a \rangle) \leq d(K + H) = d(B)$, as desired. $\qquad\square$

(3) $cl(cl(A)) = cl(A)$ for finite A.

Proof of (3). By (1) $A \subseteq cl(A)$, and by (2) $cl(A) \subseteq cl(cl(A))$. To prove the other direction assume $a \in cl(cl(A))$. Choose $a_1, \dots, a_n \in cl(A) \setminus A$ such that $a \in cl(\{a_1, \dots, a_n\} \cup A)$. Then $d(A) = d(A + \langle a_n \rangle)$. By (2) $a_1, \dots, a_{n-1} \in cl(A \cup \{a_n\})$. If we iterate the argument we get $d(A) = d(A + \langle \{a_1, \dots, a_n\} \rangle)$. Using Lemma 3.7 twice we get the desired result:

$$d(A + \langle a \rangle) \leq d(A + \langle \{a_1, \dots, a_n, a\} \rangle)$$
$$= d(A + \langle \{a_1, \dots, a_n\} \rangle) = d(A) \leq d(A + \langle a \rangle). \quad \square$$

(4) **The Steinitz Exchange Lemma**. *For finite $A \subseteq M$, $a \in cl(A \cup \{b\}) \setminus cl(A)$ implies $b \in cl(A \cup \{a\})$.*

Proof of (4). By the assumption and Lemma 3.7 we know that $d(A) < d(A + \langle a \rangle) \leq d(A + \langle \{a, b\} \rangle) = d(A + \langle b \rangle)$. Hence by Lemma 3.8 $d(\langle A \cup \{a, b\} \rangle) = d(A + \langle b \rangle) = d(A) + 1$. By assumption $d(A + \langle a \rangle) > d(A)$ and by Lemma 3.8 $d(A + \langle a \rangle) = d(A) + 1$. Both equations together imply $d(\langle A \cup \{a, b\} \rangle) = d(A + \langle a \rangle)$, as desired. $\qquad\square$

Definition. Let A be a finite subspace of M. We say that A is *n-selfsufficient* in M if $\delta(A) \leq \delta(B)$ for every finite B with $A \subseteq B$ and $dim(B/A) = n$.

Note that a subspace A is selfsufficient in M if and only if it is n-selfsufficient in M for all n. There are formulas $\varphi_{n,m}(x_1, \ldots, x_m)$ which express the property that $x_1, \ldots, x_m$ generate an n-selfsufficient subspace.

Lemma 3.10. *If A is selfsufficient in B and B is selfsufficient in M, then A is selfsufficient in M. If A is n-selfsufficient in B and B is selfsufficient in M then A is n-selfsufficient in M.*

Proof. For every $K \supseteq A$ we have to show that $\delta(A) \leq \delta(K)$. This is clear for $K \subseteq B$ and for $B \subseteq K$. Assume neither $K \subseteq B$ nor $B \subseteq K$. Then $\delta(A) \leq \delta(B \cap K)$ and $\delta(A) \leq \delta(B)$. If $\delta(K) < \delta(A)$ then $\delta(K) < \delta(B \cap K)$. By Corollary 3.5 $\delta(B + K) < \delta(B)$. Since B is *selfsufficient* in M, this is a contradiction. The same proof works for A n-selfsufficient in B. □

Lemma 3.11. *Let A be a finite subspace of B.*
  1. *If A is selfsufficient in B, then for every $K \subseteq B$ we obtain that $A \cap K$ is selfsufficient in $B \cap K = K$.*
  2. *If A is n-selfsufficient in B, then for every $K \subseteq B$ we obtain that $A \cap K$ is n-selfsufficient in $B \cap K = K$.*

Proof. To prove (1) we have to show that for every C with $A \cap K \subseteq C \subseteq K$ we have $\delta(A \cap K) \leq \delta(C)$. Note that $A \cap K = A \cap C$. Now $\delta(C) < \delta(A \cap C)$ would imply $\delta(A + C) < \delta(A)$ by Corollary 3.5. This is a contradiction to the *selfsufficiency* of A in B.

(2) follows in the same way. We have to consider C with $dim(C/A \cap C) \leq n$ only. Note that $dim((A + C)/A) = dim(C/A \cap C)$. □

4. The First Amalgamation Lemma

Let A, B, and C be structures in $\mathcal{S}$, where A is a common substructure of B and C. We define a structure $D = B \star_A C$, that we call the *free amalgam* of B and C over A: Let D be the vector space that is the free amalgam $B \oplus_A C$ of the vector spaces B and C over A in the category of vector spaces over the field with p elements. It remains to define the subspace $N(D)$ of $\wedge^2 D$. $\wedge^2 B$, $\wedge^2 C$ and $\wedge^2 A$ are subspaces of $\wedge^2 D$ with $\wedge^2 B \cap \wedge^2 C = \wedge^2 A$. Therefore $N(B)$ and $N(C)$ are subspaces of $\wedge^2 D$ with $N(B) \cap N(C) = N(A)$ (Lemma 3.3). We define $N(D)$ to be the subspace of $\wedge^2 D$ generated by $N(B) \cup N(C)$. In particular we have $\delta(D) = \delta(B) + \delta(C) - \delta(A)$. Under certain conditions we will show $(\Sigma 3)$ for D provided that A, B, and C fulfill $(\Sigma 3)$.

Let $(M, N(M))$ be a structure in $\mathcal{S}$ with $(\Sigma 3)^0$ and let $H \subseteq K$ be subspaces of M. We say that K is a *minimal extension* of H if $\delta(H) = \delta(K)$ and $\delta(L) > \delta(H)$ for every L with $H \subseteq L \subseteq K$ and $H \neq L \neq K$. We also use this notion for the induced $\mathcal{S}$-substructures. K_1 is a *realization* of the minimal extension K of H, if there is a vector space isomorphism of K onto K_1 that is the identity on H and that induces an isomorphism of the corresponding structures in $\mathcal{S}$.

Theorem 4.1 (First Amalgamation Theorem). *Assume that A, B, C are finite structures in $\mathcal{S}$ that fulfill $(\Sigma 3)$, where A is a common substructure of B and C. Suppose that A is selfsufficient in B and n-selfsufficient in C, where $2 + dim(B/A) \leq n$. Then $D = B \star_A C$ fulfills $(\Sigma 3)^0$, C is selfsufficient in D and B is n-selfsufficient*

in D. If B is a minimal extension of A that is not realized in C, or if $dim(B) = dim(A) + 1$ and $\delta(B) = \delta(A) + 1$, then D fulfills $(\Sigma 3)$.

Let us introduce the following notation. If M is an $\mathcal{S}$-structure and X is an ordered basis of the vector space M, then we call the elements $a \wedge b$ of $\wedge^2 M$ with $a < b$ the *basic commutators*. They form a basis of $\wedge^2 M$. If we speak about the basic commutators of $\wedge^2 M$ with respect to the basis X, then we assume that some ordering of X is given. If X is of the form $X_1 \ldots X_n$, then we assume that the ordering of X has the property $x < y$ for $x \in X_i$ and $y \in X_j$ with $i < j$.

First we prove some helpful lemmas. If we want to show $(\Sigma 3)$, we have to consider any $K \subseteq D$. Then $(K \cap B) \cap (K \cap C) = K \cap (B \cap C) = K \cap A$. Let Y^a be a basis for the vector space $K \cap A$, $Y^a Y^b$ be a basis for $K \cap B$, and $Y^a Y^c$ be a basis of $K \cap C$. Then $Y^b Y^c$ is linearly independent over A. If $dim(K) = dim(K \cap B) + dim(K \cap C) - dim(K \cap A)$ then $Y^a Y^b Y^c$ is a basis for K. By induction on $dim(K) - (dim(K \cap B) + dim(K \cap C) - dim(K \cap A))$ we show:

Lemma 4.2. *Let K be a finite subspace of D. Assume that $d_1, \ldots, d_m$ are linearly independent over $(K \cap B) + (K \cap C)$ such that*

$$\langle (K \cap B) \cup (K \cap C) \cup \{d_1, \ldots, d_m\} \rangle = K.$$

Then for every $U = \{u_1, \ldots, u_m\} \subseteq B$ and $V = \{v_1, \ldots, v_m\} \subseteq C$ with $u_i = d_i$ modulo C and $d_i = u_i + v_i$ for $1 \le i \le m$ we obtain that UV is linearly independent over $(K \cap B) + (K \cap C) + A$.

Proof. $m = 0$ is the base of the induction. Otherwise choose

$$H = \langle (K \cap B) \cup (K \cap C) \cup \{d_1, \ldots, d_{m-1}\} \rangle.$$

Then $dim(H) - (dim(K \cap B) + dim(K \cap C) - dim(K \cap A)) = m - 1$. We apply the induction hypothesis and obtain that $Y^b Y^c \{u_1, \ldots, u_{m-1}\}\{v_1, \ldots, v_{m-1}\}$ is linearly independent over A. If the assertion were not true, we could infer

$$\sum_{1 \le i \le m} r_i u_i + \sum_{1 \le i \le m} s_i v_i = 0 \ modulo \ \langle Y^b Y^c A \rangle$$

where $r_m \ne 0$ or $s_m \ne 0$. Assume w.l.o.g. that $s_m \ne 0$. Then there is some z in $\langle Y^c \rangle$ such that

$$\sum_{1 \le i \le m} s_i d_i + z = \sum_{1 \le i \le m} (s_i - r_i) u_i \ modulo \ \langle Y^b A \rangle.$$

It follows that there is some $h \in H$ such that $s_m d_m + h \in K \cap B$. Hence $d_m \in H$, a contradiction. $\square$

A basis $Y^a Y^b Y^c \{u_1 + v_1, \ldots, u_m + v_m\}$ of a subspace K of D as described in Lemma 4.2 is called a *suitable basis*. A suitable basis $Z^a Z^b Z^c$ of D is *consistent* with the suitable basis $Y^a Y^b Y^c \{u_1 + v_1, \ldots, u_m + v_m\}$ of K, if $Y^a \subseteq Z^a$, $Y^b \subseteq Z^b$, $Y^c \subseteq Z^c$, and the roles of $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ can be played by elements of $Z^b \backslash Y^b$ and $Z^c \backslash Y^c$ respectively. The description of K above gives us a description of $N(K) = N(D) \cap \wedge^2 K$.

Lemma 4.3. *Let $Y^a Y^b Y^c UV$ be a part of a basis of D chosen according to K as above in Lemma 4.2. Assume that there is given an order of this basis with $Y^a < Y^b < Y^c < U < V$. Then a linear combination Φ of basic commutators over $Y^a Y^b Y^c UV$ is in $N(D)$ if and only if $\Phi = \Phi_B + \Phi_C + \Phi_U + \Phi_V$, where:*

Φ_B is a linear combination of basic commutators $a \wedge b$ with $a, b \in Y^a Y^b$ and $a < b$,

Φ_C is a linear combination of basic commutators $a \wedge b$ with $a, b \in Y^a Y^c$ and $a < b$,

Φ_U is a linear combination of basic commutators $a \wedge u_j$ with $a \in Y^a$ and $u_j \in U$,

Φ_V is obtained from Φ_U by replacing u_j by v_j,

and there exists $J \in \wedge^2 A$ such that $\Phi_B + \Phi_U + J \in N(B)$ and $\Phi_C + \Phi_V - J \in N(C)$.

Proof. $\wedge^2 K$ has a basis $\{a \wedge b : a < b, a, b \in Y^a Y^b Y^c \{u_i + v_i : 1 \leq i \leq m\}\}$. Let Φ be a linear combination of such basic commutators that is in $N(D)$. With respect to any basis of D, that extends $Y^a Y^b Y^c UV$, we have $\Phi = \Phi_B + \Phi_C + \sum_i \alpha_i(a_i \wedge u_i) + \sum_i \alpha_i(a_i \wedge v_i)$, where $\Phi_B \in \wedge^2 \langle Y^a Y^b \rangle$, $\Phi_C \in \wedge^2 \langle Y^a Y^c \rangle$, and $a_i \in Y^a Y^b Y^c \{u_l + v_l : 1 \leq l \leq i\}$, because $b \wedge c$ with $b \in Y^b$ and $c \in Y^c$ cannot occur in $\Phi \in N(B) + N(C)$. By the same argument rb, rc, $r(u_i + v_i)$ with $b \in Y^b$ and $c \in Y^c$ cannot occur in the presentation with respect to the basis above of any a_j. Therefore $a_i \in \langle Y^a \rangle$. We define $\Phi_U = \sum_i \alpha_i(a_i \wedge u_i)$ and $\Phi_V = \sum_i \alpha_i(a_i \wedge v_i)$. We get $\Phi = \Phi_B + \Phi_U + \Phi_C + \Phi_V = \Psi_B + \Psi_C$, where $\Psi_B \in N(B)$ and $\Psi_C \in N(C)$. Let J be $\Psi_B - (\Phi_B + \Phi_U) = -\Psi_C + (\Phi_C + \Phi_V)$. Then $J \in \wedge^2 B \cap \wedge^2 C = \wedge^2 A$. The assertion follows. □

Lemma 4.4. *Assume that A, B, and C are finite structures in $\mathcal{S}$ with $(\Sigma 3)$, such that A is a common substructure of B and C. Let D be $B \star_A C$. Suppose that A is selfsufficient in B, and A is $(dim(B/A)+n)$-selfsufficient in C, and B is a minimal extension of A, that is not realized in C. Assume that $U = \{u_1, \ldots, u_m\} \subseteq B$ and $V = \{v_1, \ldots, v_m\} \subseteq C$, such that UV is linearly independent over A. Let $X = \{a_1, \ldots, a_k\}$ be a part of a basis of A. For $1 \leq i \leq h$ let*

$$\Phi_i = \sum_{\substack{1 \leq s \leq k \\ 1 \leq t \leq m}} r^i_{st}(a_s \wedge u_t + v_t)$$

be relations in $N(\langle X\{u_1 + v_1, \ldots, u_m + v_m\}\rangle)$ that are linearly independent over $N(\langle X \rangle)$. Then $h < m$.

Proof. Assume that $h \geq m$. By Lemma 4.3 there exists $J_i \in \wedge^2 A$ such that

$$\Psi_i = \sum_{\substack{1 \leq s \leq k \\ 1 \leq t \leq m}} r^i_{st}(a_s \wedge u_t) + J_i \in N(B),$$

$$\Delta_i = \sum_{\substack{1 \leq s \leq k \\ 1 \leq t \leq m}} r^i_{st}(a_s \wedge -v_t) + J_i \in N(C).$$

Then $\Phi_i = \Psi_i - \Delta_i$. We can write down J_i as a linear combination of basic commutators over some basis of A that extends X. Since the Φ_i's are linearly independent over $\wedge^2 \langle X \rangle$, every non-trivial linear combination of them contains $a_s \wedge u_t$ and $a_s \wedge v_t$ for some s and t. Therefore every nontrivial linear combination of the Ψ_i's and every nontrivial linear combination of the Δ_i's contains a basic commutator $a_s \wedge u_t$ and $a_s \wedge v_t$ respectively. Hence the Ψ_i's and the Δ_i's respectively are linearly independent over $\wedge^2 A$. They freely generate $N(\langle UA \rangle)$ over $N(A)$ and $N(\langle VA \rangle)$ over $N(A)$ respectively, since A is $dim(B/A)$-selfsufficient in B and C, $m \leq dim(B/A)$, and $h \geq m$. It follows that $h = m$. Therefore the vector space

isomorphism f, defined by $f(a) = a$ for $a \in A$ and $f(u_j) = -v_j$ of $\langle AU \rangle$ onto $\langle AV \rangle$ induces an isomorphism of the corresponding structures in $\mathcal{S}$. Since B is a minimal extension of A and $\delta(\langle AU \rangle) = \delta(A)$ (in B), we obtain $\langle AU \rangle = B$. Now $\langle AV \rangle$ is a realization of the minimal extension B of A in C. This contradicts the assumption of the lemma. $\qquad\square$

Lemma 4.5. *Assume that M is a structure in $\mathcal{S}$ sucht that $\delta(K) = dim(K)$ for $K \subseteq M$ with $dim(K) \leq 2$. Then $\delta(K) = dim(K)$ for $K \leq M$ with $dim(K) \leq 3$.*

Proof. Choose $K \subseteq M$ with $dim(K) = 3$. Assume that K has the basis $\{a, b, c\}$. If $\delta(K) < 3$ then there is a relation $r_1(a \wedge b) + r_2(a \wedge c) + r_3(b \wedge c)$ in $N(K)$. Assume w.l.o.g. that $r_1 \neq 0$. If $r_3 = 0$, then $r_2 \neq 0$ by assumption. But then $a \wedge (r_1 b + r_2 c) \in N(K)$, a contradiction. We obtain a similar contradiction if $r_2 = 0$. Therefore we can assume w.l.o.g. that $r_1 = 1$, $r_2 \neq 0$, and $r_3 \neq 0$. Hence $a \wedge (b + r_2 c) + r_3(b \wedge c) \in N(K)$. Choose s such that $sr_2 = r_3$ *modulo* p. Then $a \wedge (b + r_2 c) + sb \wedge (b + r_2 c) \in N(K)$ and $(a + sb) \wedge (b + r_2 c) \in N(K)$, a contradiction. $\qquad\square$

Proof of Theorem 4.1. We show $(\Sigma 3)^0$ for D and $(\Sigma 3)$ under the further assumptions. Let $K \subseteq D$. Y^a, Y^b, Y^c, U and V are chosen as above. If $dim(K) \leq 2$ then by Lemma 4.3 either $N(K) = \langle 0 \rangle$ or $K = \langle \{a, u + v\} \rangle$, where $a \in A$, $u \in B$, $v \in C$, $\{u, v\}$ is linearly independent over A, and $N(K)$ is generated by $a \wedge (u + v)$. It follows that $\delta(K) \geq 1$. This implies $(\Sigma 3)^0$ for K with $dim(K) \leq 2$. Assume that $dim(K) = 2$. If $dim(B) = dim(A) + 1$ and $\delta(B) = \delta(A) + 1$ then $dim(N(K)) = 0$, since a relation $a \wedge u + J \in N(B)$ is not possible. If B is a minimal extension of A that is not realized in C, then $dim(N(K)) = 0$ follows from Lemma 4.4. Hence the assertion is proved for all $K \subseteq M$ with $dim(K) \leq 2$.

Now we can suppose that $dim(K) \geq 3$. We choose a basis for K as above. We say that K is moderate if $U = V = \emptyset$. First we show $(\Sigma 3)$ for moderate K. Using Lemma 4.3 we prove for moderate K:

(1) $dim(N(K)) \leq dim(N(K \cap C) + dim(N((K \cap B) + A)) - dim(N(A))$.

Proof of (1). $N(K)$ is a subspace of $\wedge^2 K = \wedge^2((K \cap B) + (K \cap C))$. We construct a basis of $N(K)$. First we choose $\Phi_1, \dots, \Phi_s$ as a basis of $N(K \cap C)$. Let $\Delta_1, \dots, \Delta_r$ be linear combinations of basic commutators in $N(K)$ such that their images in $N(K)/N(K \cap C)$ are a basis of this vector space. By Lemma 4.3 $\Delta_i = \Delta_i^B + J_i + \Delta_i^C - J_i$ where $\Delta_i^B \in \wedge^2(K \cap B)$, $\Delta_i^C \in \wedge^2(K \cap B)$, $J_i \in \wedge^2 A$, $\Delta_i^B + J_i \in N(B)$, and $\Delta_i^C - J_i \in N(C)$. Δ_i^B is a linear combination of basic commutators over $Y^a Y^b$. Then every non-trivial linear combination Δ of the Δ_i^B contains a basic commutator $w \wedge y$ with $y \in Y^b$, since otherwise the corresponding linear combination of the Δ_i's would be in $N(K \cap C)$. Hence the $\Delta_i^B + J_i \in N((K \cap B) + A)$ are linearly independent over $\wedge^2 A$ and therefore over $N(A)$. $\qquad\square$

Note that $dim(K) = dim(K \cap C) + dim((K \cap B) + A) - dim(A)$. Therefore (1) implies

(2) $\delta(K) \geq \delta(K \cap C) + \delta((K \cap B) + A) - \delta(A)$.

Since A is selfsufficient in B we obtain $\delta((K \cap B) + A) \geq \delta(A)$, and therefore

(3) $\delta(K) \geq \delta(K \cap C)$.

$(\Sigma 3)^0$ follows for moderate K and $(\Sigma 3)$ for moderate K with $dim(K \cap C) \geq 3$.

If $dim(B) = dim(A) + 1$ and $\delta(B) = \delta(A) + 1$, then either $K \subseteq C$ or $\delta(K) = \delta(K \cap C) + 1$ by (2). The assertion follows from (Σ3) for C. Now we can assume that B is a minimal extension of A, that is not realized in C. Similarly as in (2) we obtain

(4) $\delta(K) \geq \delta(K \cap B) + \delta((K \cap C) + A) - \delta(A)$.

It suffices to consider the cases where $|Y^c| \leq 2$. Since $dim((K \cap C) + A/A) = |Y^c| \leq 2$, the 2-selfsufficiency of A in C implies

(5) $\delta(K) \geq \delta(K \cap B)$, if $|Y^c| \leq 2$.

The assertion follows if $dim(K \cap B) \geq 3$. Hence we can suppose that $dim(K \cap C) < 3$ and $dim(K \cap B) < 3$. At the beginning (Σ3) was shown for all K with $dim(K) \leq 2$. By Lemma 4.5 it is true for K with $dim(K) \leq 3$. We can assume that $dim(K) > 3$. The only possibility to fulfill all these conditions is $|Y^a| = 0$, $|Y^b| = |Y^c| = 2$. In this case we show $dim(N(K)) \leq 1$. Let Y^b be $\{y_1, y_2\}$ and Y^c be $\{z_1, z_2\}$. By (Σ3) for B and C, $y_1 \wedge y_2$ and $z_1 \wedge z_2$ are not in $N(K)$. Therefore by Lemma 4.3 every element in $N(K)$ has the form $r_1(y_1 \wedge y_2) + r_2(z_1 \wedge z_2)$ with $r_1 \neq 0$ and $r_2 \neq 0$. There are not two linearly independent elements of this form in $N(K)$, since their existence would imply $y_1 \wedge y_2 \in N(K)$. It follows that $dim(N(K)) \leq 1$, as desired. (Σ3) is shown for moderate K.

Now we assume that K is not moderate. Remember that by assumption $dim(K) \geq 3$. Let K^- be $(K \cap B) + (K \cap C) = \langle Y^a Y^b Y^c \rangle$. K^- is moderate. By definition $N(K^-) \subseteq N(K)$. We use a similar proof as in the moderate case. First we show

(6) $dim(N(K)) \leq dim(N(K \cap C)) + dim(N(\langle Y^b U A \rangle)) - dim(N(A))$.

Proof of (6). Let $\Phi_1, \ldots, \Phi_s$ be a basis of $N(K \cap C)$. Let $\Delta_1, \ldots, \Delta_r$ in $N(K^-)$ be a basis of $N(K^-)/N(K \cap C)$. In the proof of (1) it is shown that $\Delta_1^+ = \Delta_1^B + J_1, \ldots, \Delta_r^+ = \Delta_r^+ + J_r \in N(\langle Y^b A \rangle)$ are linearly independent over $\wedge^2 A$. Now we choose $\Psi_1, \ldots, \Psi_t$ in $N(K)$ such that their images in $N(K)/N(K^-)$ are a basis of this vector space. Then $\{\Phi_1, \ldots, \Phi_s, \Delta_1, \ldots, \Delta_r, \Psi_1, \ldots, \Psi_t\}$ is a basis of $N(K)$. According to Lemma 4.3 let $\Psi_i = \Psi_i^B + \Psi_i^C + \Psi_i^U + \Psi_i^V$ and $I_i \in \wedge^2 A$ with $\Psi_i^B + \Psi_i^U + I_i \in N(B)$ and $\Psi_i^C + \Psi_i^V - I_i \in N(C)$. Then every non-trivial linear combination of the Ψ_i's contains some $a \wedge u_j$ with $a \in Y^a$ and $u_j \in U$. Therefore the $\Psi_i^+ = \Psi_i^B + \Psi_i^U + I_i \in N(B)$ are linearly independent over $\langle \wedge^2 A \cup \{\Delta_1^+, \ldots, \Delta_r^+\} \rangle$. Claim (6) follows, since $\Delta_1^+, \ldots, \Delta_r^+, \Psi_1^+, \ldots, \Psi_t^+$ are elements of $N(\langle Y^b U A \rangle)$ that are linearly independent over $\wedge^2 A$. □

As above it follows that

(7) $\delta(K) \geq \delta(K \cap C) + \delta(\langle Y^b U A \rangle) - \delta(A)$.

By the selfsufficiency of A in B we obtain that $\delta(\langle Y^b U A \rangle) - \delta(A) \geq 0$ and therefore

(8) $\delta(K) \geq \delta(K \cap C)$.

(Σ3)^0 follows for all K, as does (Σ3) for all K with $dim(K \cap C) \geq 3$. Note that we have assumed that $dim(K) \geq 3$ and K is not moderate. If $dim(B) = dim(A) + 1$ and $\delta(B) = \delta(A) + 1$, then $dim(K \cap C) = dim(K) - 1$, $|Y^b| = 0$, $|U| = 1$, and $\delta(\langle Y^b U A \rangle) - \delta(A) = 1$. Therefore ($\Sigma$3) follows from (7) in this case.

Now we assume that B is a minimal extension of A that is not realized in C. The assertion is proved for $dim(K \cap C) \geq 3$. If $dim(K \cap C) < 3$, then we can show analogously as above that $\delta(K) \geq \delta(K \cap B)$. We use the 2-selfsufficiency of A in C. Therefore we can assume now that $dim(K \cap C) < 3$ and $dim(K \cap B) < 3$. The following cases are possible: $|Y^a| = 0$, $|Y^a| = 1$, and $|Y^a| = 2$.

Case $|Y^a| = 0$: By Lemma 4.3 $N(K) = N(K^-)$. Therefore $\delta(K) = dim(K) - dim(K^-) + \delta(K^-)$. Since the assertion is already proved for the moderate K^-, it follows for K.

Case $|Y^a| = 1$: Since $dim(K \cap C) < 3$ and $dim(K \cap B) < 3$, $|Y^b|, |Y^c| \le 1$. Let Φ be any non-trivial relation in $N(K)$. Note that $N(\langle Y^a \rangle) = 0$. If $Y^a = \{a\}$, then by Lemma 4.3 $\Phi = a \wedge (u + v)$, where $u \in \langle Y^b U \rangle$ and $v \in \langle Y^c V \rangle$. $u = 0$ or $v = 0$ is impossible by $(\Sigma 3)$ for B and C. By Lemma 4.4 $dim(N(\langle \{a, u + v\} \rangle)) = 0$ and therefore $dim(N(K)) = 0$. Hence $\delta(K) = dim(K)$, as desired.

Case $|Y^a| = 2$: By assumption $|Y^b| = |Y^c| = 0$. Let Y^a be $\{a_1, a_2\}$, $|U| = l$. We suppose that there are at least l relations Φ_i in $N(K)$ that are linearly independent over $N(\langle Y^a \rangle) = \langle 0 \rangle$, and show a contradiction. By Lemma 4.3:

(9) $\Phi_i = \sum_{\substack{s=1,2 \\ 1 \le t \le l}} r^i_{st}(a_s \wedge (u_t + v_t)) + r^i_0(a_1 \wedge a_2)$.

We show that it is possible to replace every u_t by another element u_t^+ of its coset modulo $\langle Y^a \rangle$, such that the representation of (9) with respect to this basis $Y^a U^+ V$ has no summands $r^i_0(a_1 \wedge a_2)$. Then we apply Lemma 4.4 and obtain a contradiction. It follows that $dim(N(\langle Y^a \{u_1 + v_1, \dots, u_l + v_l\} \rangle)) \le l - 1$. Therefore $\delta(K) \ge 3$, as desired.

To get the desired basis $Y^a U^+ V$ we start with Φ_1. Since $\Phi_1 \notin N(\langle Y^a \rangle)$ there are s_1 and t_1 such that $r^1_{s_1 t_1} \ne 0$. For each i with $2 \le i \le l$ we subtract a suitable $q_i \Phi_1$ from Φ_i. Therefore we can assume w.l.o.g. that $r^i_{s_1 t_1} = 0$ for $2 \le i \le l$. Using the linear independence of $\Phi_1, \dots, \Phi_l$ over $\wedge^2 A$ we iterate this procedure. Therefore we can assume w.l.o.g. that there are pairwise distinct pairs (s_i, t_i) such that $r^i_{s_i t_i} \ne 0$ and $r^j_{s_i t_i} = 0$ for $i \ne j$. For each t with $1 \le t \le l$ we define u_t^+:

(i) If there is no i with $t_i = t$, then $u_t^+ = u_t$.
(ii) If there is exactly one i with $t_i = t$, then $u_t^+ = u_t + (-1)^{3-s_i}(r^i_0 / r^i_{s_i t_i}) a_{3-s_i}$.
(iii) If $t_i = t_j = t$ for $i \ne j$, then

$$u_t^+ = u_t + (-1)^{3-s_i}(r^i_0 / r^i_{s_i t}) a_{3-s_i} + (-1)^{3-s_j}(r^j_0 / r^j_{s_j t}) a_{3-s_j}.$$

Note that in case (iii) $s_i = 3 - s_j$ and $s_j = 3 - s_i$. Now we write down (9) with respect to the basis $Y^a U^+ V$: Let I be the set of all t with $1 \le t \le l$ and $t \ne t_j$ for all j. Let J_i be the set of all t with $1 \le t \le l$ and $t = t_j$ for exactly one j, and this j is not i.

$$\begin{aligned}
\Phi_i &= \sum_{\substack{s=1,2 \\ t \in I}} r^i_{st}\left(a_s \wedge (u_t + v_t)\right) \\
&\quad + r^i_{s_i t_i}\left(a_{s_i} \wedge (u_{t_i} + v_{t_i} + (-1)^{3-s_i}(r^i_0 / r^i_{s_i t_i}) a_{3-s_i})\right) \\
&\quad + r^i_{3-s_i t_i}\left(a_{3-s_i} \wedge (u_{t_i} + v_{t_i} + (-1)^{3-s_i}(r^i_0 / r^i_{s_i t_i}) a_{3-s_i})\right) \\
&\quad + \sum_{t_j \in J_i} r^i_{3-s_j t_j}\left(a_{3-s_j} \wedge (u_t + v_t + (-1)^{3-s_j}(r^j_0 / r^j_{s_j t_j}) a_{3-s_j})\right) \\
&= \sum_{\substack{s=1,2 \\ 1 \le t \le l}} r^i_{st}\left(a_s \wedge (u_t^+ + v_t)\right).
\end{aligned}$$

Note that $r^i_{s_j t_j} = 0$ for $i \ne j$. If there is some $j \ne i$ such that $t_j = t_i$, then $r^i_{3-s_i t_i} = r^i_{s_j t_j} = 0$. This finishes the proof of $(\Sigma 3)$ for the situation where B is a minimal extension of A that is not realized in C.

Next we show that C is selfsufficient in D. If $C \subseteq K$, then K is moderate, $K \cap C = C$, and by (3) $\delta(K) \geq \delta(K \cap C) = \delta(C)$, as desired. Finally we prove that B is n-selfsufficient in D. If $B \subseteq K$ and $dim(K/B) \leq n$, then K is moderate, $K \cap B = B$ and by (4) $\delta(K) \geq \delta(B) + \delta(K \cap C) - \delta(A)$. Since $dim((K \cap C)/A) = dim(K/B) \leq n$, it follows that $\delta(K) \geq \delta(B)$ by the n-selfsufficiency of A in C, as desired. $\square$

5. Minimal Extensions

Let D be $B \star_A C$ as in Theorem 4.1, where A, B, and C are finite structures in $\mathcal{S}$ that satisfy $(\Sigma 3)$, A is a common substructure of B and C, A is selfsufficient in B and $(\dim(B/A)+n)$-selfsufficient in C. Furthermore we suppose that B is a minimal extension of A that is not realized in C. By Theorem 4.1 D satisfies $(\Sigma 3)$, C is selfsufficient in D, and B is n-selfsufficient in D. We consider a minimal extension K of H (both finite) in D. As defined above, this means that $\delta(H) = \delta(K)$ but $\delta(L) > \delta(H)$ for every L with $H \subseteq L \subseteq K$ and $H \neq L \neq K$. According to Lemma 4.2 we choose a suitable basis for H in the following way:

$(*)$

Let Y^a be a basis for $H \cap A$.

Let Y^b be a sequence such that $Y^a Y^b$ is a basis for $H \cap B$.

Let Y^c be a sequence such that $Y^a Y^c$ is a basis for $H \cap C$.

Let Y^d be a basis of H over $(H \cap B) + (H \cap C)$.

Then $Y^a Y^b Y^c Y^d$ is a suitable basis of H.

Now we start with $Y^a Y^b Y^c Y^d$, which we will extend to a suitable basis for K. First we choose X^a such that $Y^a X^a$ is a basis for $K \cap A$. Let X^{by} be a basis of $K \cap B \cap (H + C)$ over $(H \cap B) + (K \cap A)$. For every $u \in X^{by}$ we choose a $v \in C$ such that $u + v \in H$. v exists by definition. It follows that $v \in C \cap K$. v is uniquely determined modulo $(H \cap C) + (K \cap A)$. Let X^{cy} be the set of these v's. Then X^{cy} is a basis of $K \cap C \cap (H + B)$ over $(H \cap C) + (K \cap A)$. By construction we obtain $|X^{by}| = |X^{cy}|$. For every $u \in X^{by}$ there is exactely one $v \in X^{cy}$ such that $u + v \in H$, and for every $v \in X^{cy}$ there is exactely one $u \in X^{by}$ with $u + v \in H$. Then we choose X^{bx} and X^{cx} such that $Y^a X^a Y^b X^{by} X^{bx}$ is a basis for $K \cap B$ and $Y^a X^a Y^c X^{cy} X^{cx}$ is a basis for $K \cap C$.

Now $Y^a X^a Y^b X^{by} X^{bx} Y^c X^{cy} X^{cx}$ is a basis for $(K \cap B) + (K \cap C)$. It can be extended to a basis of K. First we look for a basis of H over $(K \cap B) + (K \cap C)$ and then we extend it to a basis of K. By Lemma 4.2 there are

$$Y^{be} = \{u_1, \ldots, u_l\}, \qquad Y^{ce} = \{v_1, \ldots, v_l\},$$
$$X^{be} = \{t_1, \ldots, t_k\}, \qquad X^{ce} = \{w_1, \ldots, w_k\}$$

with the following properties:

• $Y^a X^a Y^b X^{by} X^{bx} Y^{be} X^{be}$ is part of a basis of B,

• $Y^a X^a Y^c X^{cy} X^{cx} Y^{ce} X^{ce}$ is part of a basis of C,

• $Y^b X^{by} X^{bx} Y^{be} X^{be} Y^c X^{cy} X^{cx} Y^{ce} X^{ce}$ is linearly independent over A

• $Y^e = \{u_1 + v_1, \ldots, u_l + v_l\} \subseteq H$ is a basis of H modulo $(K \cap B) + (K \cap C)$, and

• $Y^a Y^b Y^c Y^e X^a X^{by} X^{bx} X^{cy} X^{cx} X^e$ is a suitable basis of K,

where $X^e = \{t_1 + w_1, \ldots, t_k + w_k\}$. We call it a *suitable basis of K that respects H*.

We also say that $X^a X^{by} X^{bx} X^{cx} X^e$ or $X^a X^{bx} X^{cy} X^{cx} X^e$ respectively is a suitable extension of $Y^a Y^b Y^c Y^d$ for K. All such terminology refers to the situation described above.

Lemma 5.1. *Assume that K is a minimal extension of H in D.*

1. *If $K+C \neq H+C$ and $K\cap C \neq H\cap C$, then $\delta((K+C)\cap B) < \delta((H+C)\cap B)$.*
2. *If $K+B \neq H+B$ and $K\cap B \neq H\cap B$, then $\delta((K+B)\cap C) < \delta((H+B)\cap C)$.*
3. *If $K\cap C = H\cap C$, $K\cap B \neq H\cap B$, and $K \neq H+(K\cap B)$, then $\delta((K+B)\cap C) < \delta((H+B)\cap C)$.*
4. *If $K\cap B = H\cap B$, $K\cap C \neq H\cap C$, and $K \neq H+(K\cap C)$, then $\delta((K+C)\cap B) < \delta((H+C)\cap B)$.*
5. *If $K \cap C = H \cap C$, then $\delta((K+C)\cap B) \leq \delta((H+C)\cap B)$.*
6. *If $K \cap B = H \cap B$, then $\delta((K+B)\cap C) \leq \delta((H+B)\cap C)$.*

Proof. Assume that $Y = Y^a Y^b Y^c Y^d$ is a suitable basis of H, and

$$X = X^a X^{by} X^{bx} X^{cx} X^e \quad \text{or} \quad X = X^a X^{bx} X^{cy} X^{cx} X^e$$

respectively gives us a suitable extension of Y. Let

$$Y^a Y^b Y^c Y^e X^a X^{by} X^{bx} X^{cy} X^{cx} X^e$$

be the corresponding suitable basis for K, where $Y^e = \{u_1 + v_1, \ldots, u_l + v_l\}$ and $X^e = \{t_1 + w_1, \ldots, t_k + w_k\}$. Using this suitable basis, we can formulate the assertions of the lemma in the following way:

1. If $\emptyset \neq X^{bx} X^e \neq X$, then

$$\dim(N(\langle X^{bx}\{t_1, \ldots, t_k\} Y^b X^{by}\{u_1, \ldots, u_l\} A\rangle))$$
$$- \dim(N(\langle Y^b X^{by}\{u_1, \ldots, u_l\} A\rangle)) > |X^{bx}| + k.$$

2. If $\emptyset \neq X^{cx} X^e \neq X$, then

$$\dim(N(\langle X^{cx}\{w_1, \ldots, w_k\} Y^c X^{cy}\{v_1, \ldots, v_l\} A\rangle))$$
$$- \dim(N(\langle Y^c X^{cy}\{v_1, \ldots, v_l\} A\rangle)) > |X^{cx}| + k.$$

3. If $X^{bx} X^e = X$, $X^{bx} \neq \emptyset$, and $X^e \neq \emptyset$, then

$$\dim(N(\langle \{w_1, \ldots, w_k\} Y^c \{v_1, \ldots, v_l\} A\rangle))$$
$$- \dim(N(\langle Y^c \{v_1, \ldots, v_l\} A\rangle)) > k.$$

4. If $X^{cx} X^e = X$, $X^{cx} \neq \emptyset$, and $X^e \neq \emptyset$, then

$$\dim(N(\langle \{t_1, \ldots, t_k\} Y^b \{u_1, \ldots, u_l\} A\rangle))$$
$$- \dim(N(\langle Y^b \{u_1, \ldots, u_l\} A\rangle)) > k.$$

5. If $X^{bx} X^e = X$, then

$$\dim(N(\langle X^{bx}\{t_1, \ldots, t_k\} Y^b \{u_1, \ldots, u_l\} A\rangle))$$
$$- \dim(N(\langle Y^b \{u_1, \ldots, u_l\} A\rangle)) \geq |X|.$$

6. If $X^{cx} X^e = X$, then

$$\dim(N(\langle X^{cx}\{w_1, \ldots, w_k\} Y^c \{v_1, \ldots, v_l\} A\rangle))$$
$$- \dim(N(\langle Y^c \{v_1, \ldots, v_l\} A\rangle)) \geq |X|.$$

We have rephrased the statements of the lemma in the form above since the proofs use a basic commutator argument.

ad (1)   We assume that $\{u_1^+, \ldots, u_m^+\} = X^{by}\{u_1, \ldots, u_l\}$ and $\{v_1^+, \ldots, v_m^+\} = X^{cy}\{v_1, \ldots, v_l\}$. By the minimality of K over H and the assumption we have $\dim(N(\langle XY\rangle)) - \dim(N(\langle X^{cx}X^{by}X^aY\rangle)) > |X^{bx}| + k$. Let $\Phi_1, \ldots, \Phi_r$ be a basis of $N(\langle (X\setminus X^e)Y\rangle)$ over $N(\langle (X\setminus X^{bx}X^e)Y\rangle)$. Let $\Delta_1, \ldots, \Delta_s$ be a basis of $N(\langle XY\rangle)$ over $N(\langle (X\setminus X^e)Y\rangle)$. Then $r + s > |X^{bx}| + k$. First we describe the Φ_i's and Δ_i's as linear combinations of basic commutators with respect to

$$Y^aY^bY^c\{u_1^+, \ldots, u_m^+\}\{v_1^+, \ldots, v_m^+\}X^aX^{bx}X^{cx}\{t_1, \ldots, t_k\}\{w_1, \ldots, w_k\}$$

This set is part of a basis of D. By Lemma 4.3 $\Phi_i = \Phi_i^b + \Phi_i^c$, where Φ_i^b is a linear combination of basic commutators over

$$X^aX^{bx}\{u_1^+, \ldots, u_m^+\}Y^aY^b,$$

and Φ_i^c is a linear combination of basic commutators over

$$X^aX^{cx}\{v_1^+, \ldots, v_m^+\}Y^aY^c,$$

and there exists some $I_i \in \wedge^2 A$, such that $\Phi_i^b + I_i \in N(B)$ and $\Phi_i^c - I_i \in N(C)$.

By assumption every non-trivial combination of the Φ_i's contains a basic commutator $z \wedge b_x$, where $b_x \in X^{bx}$. Therefore the $(\Phi_i^b + I_i)$'s are linearly independent over $N(\langle Y^b\{u_1^+, \ldots, u_m^+\}A\rangle)$.

Again by Lemma 4.3: $\Delta_i = \Delta_i^b + \Delta_i^c + \Delta_i^{be} + \Delta_i^{ce}$, where Δ_i^b is a linear combination of basic commutators over $X^aX^{bx}Y^aY^b\{u_1^+, \ldots, u_m^+\}$, Δ_i^c is a linear combination of basic commutators over $X^aX^{cx}Y^aY^c\{v_1^+, \ldots, v_m^+\}$, Δ_i^{be} is a linear combination of basic commutators $a \wedge t_i$ where $a \in Y^aX^a$, Δ_i^{ce} is obtained from Δ_i^{be} by replacing t_i by w_i, and there exists some $J_i \in \wedge^2 A$ with $\Delta_i^b + \Delta_i^{be} + J_i \in N(B)$ and $\Delta_i^c + \Delta_i^{ce} - J_i \in N(C)$.

Since the Δ_i's are linearly independent over $N(\langle (X\setminus X^e)Y\rangle)$, every non-trivial linear combination of them contains some basic commutator $a \wedge t_i$. It follows that $\{\Phi_i^b + I_i : 1 \le i \le r\} \cup \{\Delta_i^{be} + \Delta_i^b + J_i : 1 \le i \le s\}$ is linearly independent over $N(\langle Y^b\{u_1^+, \ldots, u_m^+\}A\rangle)$, as desired.

ad (2)   Obviously (2) is proved analogously to (1).

ad (5)   Also for (5) we use the method of (1). For the start, minimality gives $\dim(N(\langle XY\rangle)) - \dim(N(\langle Y\rangle)) = |X|$.

ad (6)   Again (6) is shown as (5).

ad (3)   We use the notation of (1). By the minimality we get $s > |X^e| = k$. The linear independence of $\Delta_1, \ldots, \Delta_s$ over $N(\langle YX^{bx}\rangle)$ implies that every non-trivial linear combination of them contains the basic commutators $a \wedge t_i$ and $a \wedge w_i$ for some i and $a \in Y^aX^a$. Therefore the $(\Delta_i^c + \Delta^{ce} - J_i)$'s are linearly independent over $N(\langle Y^c\{v_1^+, \ldots, v_m^+\}A\rangle)$, as desired.

ad (4)   We show (4) similarly as (3). $\square$

Lemma 5.2. *Let K be a minimal extension of H in D. Assume that $H \subseteq A$. Let $K_1, \ldots, K_h$ be a set of realizations of K over H. Suppose that there is no realization K_i of this minimal extension over H with $(K_i \cap B) + (K_i \cap C) = H$. Then either all realizations K_i are in B or all realizations K_i are in C. In the first case $\langle AK_i\rangle = B$ or $K_i \subseteq A$.*

Proof. If for some i with $1 \le i \le h$, K_i is contained neither in B nor C, then either $K_i \cap C \ne H$ and 5.1 (1) yields $\delta((K_i + C) \cap B) < \delta(A)$, contradicting the

selfsufficiency of A in B, or else $K_i \cap C = H$ and 5.1 (3) yields $\delta((K_i + B) \cap C) < \delta(A)$, which contradicts the $(\dim(B/A))$-selfsufficiency of A in C, as $K_i \cap C \subseteq A$.

Thus for each i we have $K_i \subseteq B$ or $K_i \subseteq C$. If $K_i \subseteq B$ and K_i is not contained in A, then again by 5.1 (1) we have $K_i \cap C = H$, and by 5.1 (5) $\delta(\langle AK_i \rangle) \le \delta(A)$, forcing $\langle AK_i \rangle = B$ by the minimality of B over A.

Now suppose toward a contradiction that $\langle AK_i \rangle = B$ and that for some j $K_j \not\le B$. Then $K_j \le C$, and by 5.1 (2) and the $(\dim(B/A))$-selfsufficiency of A in C we find that $K_j \cap B = H$, and then by 5.1 (6) we get $\delta(\langle K_j A \rangle) = \delta(A)$. Let $\Phi_1, \dots, \Phi_r$ be a basis of $N(K_i)$ modulo $N(H)$, where $r = \dim(K/H)$. Then $\Phi_1, \dots, \Phi_r$ are linearly independent modulo $\wedge^2 A$. Let f be an isomorphism of K_i with K_j over H. Then $f(\Phi_1), \dots, f(\Phi_r)$ are linearly independent over $\wedge^2 H$ and hence over $\wedge^2 A$. As $\delta(\langle K_j A \rangle) = \delta(A)$, we find that $f(\Phi_1), \dots, f(\Phi_r)$ form a basis for $N(\langle AK_j \rangle)$ over $N(A)$, and hence f extends to an isomorphic embedding of $\langle AK_i \rangle = B$ into C, a contradiction. $\square$

Lemma 5.3. *Assume that M is a structure in $\mathcal{S}$ that satisfies $(\Sigma 3)$. Let $K_1, \dots, K_h$ be realizations of the minimal extension K_1 of H, where H is selfsufficient in M. Furthermore suppose that $K_i \not\subseteq \langle \bigcup_{1 \le j < i} K_j \rangle$ for every i with $1 \le i \le h$. Then for every i we have: $K_1, \dots, K_i$ are linearly independent over H and $\delta(\langle \bigcup_{1 \le j \le i} K_j \rangle) = \delta(H)$.*

Proof. We use induction on i. For $i = 1$ the assertion is clear. Assume that the assertion is proved for all $j < i$. Then $\delta(\langle \bigcup_{1 \le j < i} K_j \rangle) = \delta(H)$ and $K_1, \dots, K_{i-1}$ are linearly independent over H. By Lemma 3.4

$$\delta(\langle \bigcup_{1 \le j \le i} K_j \rangle) \le \delta(\langle \bigcup_{1 \le j < i} K_j \rangle) + \delta(K_i) - \delta(K_i \cap \langle \bigcup_{1 \le j < i} K_j \rangle).$$

By induction and the assumption of the lemma we obtain

$$(1) \qquad \delta(\langle \bigcup_{1 \le j \le i} K_j \rangle) \le \delta(H) + \delta(H) - \delta(K_i \cap \langle \bigcup_{1 \le j < i} K_j \rangle)$$

Obviously $H \subseteq (K_i \cap \langle \bigcup_{1 \le j < i} K_j \rangle) \subseteq K_i$. Let us assume that $K_1, \dots, K_i$ are not linearly independent over H. This and the linear independence of $K_1, \dots, K_{i-1}$ over H imply the first of the following two inequations: $H \ne K_i \cap \langle \bigcup_{1 \le j < i} K_j \rangle \ne K_i$. The second is the assumption of the lemma. By the minimality of K_i over H, $\delta(K_i \cap \langle \bigcup_{1 \le j < i} K_j \rangle) > \delta(H)$. By (1) $\delta(\langle \bigcup_{1 \le j \le i} K_j \rangle) < \delta(H)$, a contradiction to the selfsufficiency of H in M. Hence $K_1, \dots, K_i$ are linearly independent over H and we have $K_i \cap \langle \bigcup_{1 \le j < i} K_j \rangle = H$. By (1) $\delta(\langle \bigcup_{1 \le j \le i} K_j \rangle) \le \delta(H)$, and by the selfsufficiency of H in M we get equality, as desired. $\square$

6. The Second Amalgamation Theorem

Let M be any structure in $\mathcal{S}$ that fulfills $(\Sigma 3)$. Assume that $H \subseteq K \subseteq M$, H and K are finite subspaces, and K is a minimal extension of H. Suppose that $K_1, \dots, K_h$ is a sequence of realizations of $K = K_1$ over H. Let X_i be a basis of K_i over H. We say that $K_1, \dots, K_h$ (or $X_1, \dots, X_h$ respectively) *are in free composition* over H, if $X_1, \dots, X_h$ are linearly independent over H and if

$$\delta(\langle \bigcup_{1 \le j \le h} K_j \rangle) = \delta(H).$$

If H is selfsufficient in M and $K_i \nsubseteq \langle \bigcup_{1 \le j < i} K_j \rangle$ for all $1 \le i \le h$, then $K_1, \dots, K_h$ are in free composition over H, by Lemma 5.3. Let X_i be $\{x_1^i, \dots, x_r^i\}$ for $1 \le i \le h$. We call $X_1, \dots, X_h$ a sequence of *compatible* bases, if the map f_k defined by $f_k(x_j^1) = x_j^k$, and $f(y) = y$ for $y \in H$ induces an isomorphism of the $\mathcal{S}$ - structures K_1 onto K_k over H. Of course a compatible system of bases $X_1, \dots, X_h$ over H exists for every sequence of realizations $K_1, \dots, K_h$ of a minimal extension K_1 over H. Suppose such a system is given. Let Y be a basis of H. Let Φ be any element of $\wedge^2 K_1$, presented as a linear combination of basic commutators over $Y X_1$. Then $f_k(\Phi)$ denotes its image in $\wedge^2 K_k$, replacing each x_j^1 by x_j^k.

Lemma 6.1. *Assume that K_1 is a minimal extension of H in M. Let $X_1, \dots, X_h$ be a sequence of compatible bases of realizations $K_1, \dots, K_h$ of K_1 over H that are in free composition over H. Let σ be any permutation of $1, \dots, h$.*

1. *For every i with $1 \le i \le h$, we have that $\langle \bigcup_{1 \le j \le i} K_{\sigma(j)} \rangle$ is a minimal extension of $\langle \bigcup_{1 \le j < i} K_{\sigma(j)} \rangle$.*
2. *Let f_σ be the map $f_\sigma(x_j^i) = x_j^{\sigma(i)}$ and $f_\sigma(y) = y$ for $y \in H$. Then f_σ can be extended to an isomorphism of $\langle H X_1 \dots X_h \rangle$ onto $\langle H X_1 \dots X_h \rangle$ that induces an isomorphism of the corresponding structures in $\mathcal{S}$.*

Proof. As above let Y be a basis of H. Let us consider all permutations σ of $1, \dots, h$ and all i with $1 \le i \le h$. By the minimality of K_1 over H there are $\Phi_1, \dots, \Phi_r$ $(r = |X|)$ in $N(\langle Y X_1 \rangle)$ such that $\Phi_1, \dots, \Phi_r$ freely generate $N(\langle Y X_1 \rangle)$ over $N(H)$. We consider $\Phi_1, \dots, \Phi_r$ as linear combinations of basic commutators over $Y X_1$. Then every non-trivial linear combination of them contains some $z \wedge x_j^1$ for some $1 \le j \le r$ and $z \in Y X_1$. Hence $f_{\sigma(i)}(\Phi_1), \dots, f_{\sigma(i)}(\Phi_r)$ in $N(\langle Y X_{\sigma(i)} \rangle)$ are linearly independent over $N(H)$ and also in $N(\langle Y \bigcup_{1 \le j \le i} X_{\sigma(j)} \rangle)$ over $N(\langle Y \bigcup_{1 \le j < i} X_{\sigma(j)} \rangle)$. From this and $\delta(\sum_{1 \le j \le h} K_j) = \delta(H)$ it follows that $\delta(\langle Y \bigcup_{1 \le j \le i} X_{\sigma(j)} \rangle) = \delta(H)$. Hence $f_{\sigma(i)}(\Phi_1), \dots, f_{\sigma(i)}(\Phi_r)$ freely generate $N(\langle Y \bigcup_{1 \le j \le i} X_{\sigma(j)} \rangle)$ over $N(\langle Y \bigcup_{1 \le j < i} X_{\sigma(j)} \rangle)$. This implies both assertions of the lemma. $\square$

As above, let $K_1, \dots, K_h$ be a sequence of realizations of the minimal extension K_1 of H in M. We assume that they are in free composition over H. Again let $X_j = \{x_1^j, \dots, x_r^j\}$ for $1 \le j \le h$ be a system of compatible bases. Let σ be any permutation of $1, \dots, h$. As above in Lemma 6.1 let f_σ be the corresponding automorphism of the $\mathcal{S}$-structure generated by $H X_1 \dots X_h$ in M. We write f_i instead of f_σ, if σ is the permutation that only exchanges 1 and i. Let $\Phi_1, \dots, \Phi_r$ be free generators of $N(\langle H X_1 \rangle)$ over $N(H)$. Then $f_i(\Phi_1), \dots, f_i(\Phi_r)$ freely generate $N(H X_1 X_i)$ over $N(H X_1)$. But also $f_i(\Phi_1) - \Phi_1, \dots, f_i(\Phi_r) - \Phi_r$ do this. By definition $f_i(\Phi_j) - \Phi_j$ for $1 < i \le h$ has the following form, where $0 \le \alpha_{ts}^j < p$ and $y_t^j \in K$:

$$\sum_{s < t} \alpha_{st}^j \left((x_s^i \wedge x_t^i) - (x_s^1 \wedge x_t^1) \right) + \sum_t (y_t^j \wedge (x_t^i - x_t^1)).$$

Note that α_{st}^j and y_t^j do not depend on i. If we apply a suitable permutation to the elements of X_j uniformly for all j, then we find some k $(1 \le k \le r)$ such that $\alpha_{ts_1}^j = 0 = \alpha_{s_2 t}^j$ for all j and all s_1, s_2 if and only if $k < t$. We introduce the following notation:

For $1 \leq i \leq h$, $X_i^+ = \{x_1^i, \ldots, x_k^i\}$ and for $2 \leq i \leq h$, $X_i^- = \{z_{k+1}^i, \ldots, z_r^i\}$, where $z_t^i = x_t^i - x_t^1$. We choose a linearly independent set $Y \subseteq H$ such that

$$\langle Y \rangle = \langle \{ y_t^j : 1 \leq t \leq r, 1 \leq j \leq r \} \rangle$$

Then the sequence $Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ has the following properties:

(S1): For $2 \leq i \leq h$ the vector space $\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle$ is a minimal extension of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i-1}^+ X_{i-1}^- \rangle$. Also $N(\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle)$ is freely generated by $\Phi_1^i, \ldots, \Phi_r^i$ over $N(\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i-1}^+ X_{i-1}^- \rangle)$, where Φ_j^i is of the form

$$\sum_{1 \leq s < t \leq k} \alpha_{st}^j \left((x_s^i \wedge x_t^i) - (x_s^1 \wedge x_t^1) \right) + \sum_{1 \leq t \leq k} (y_t^j \wedge (x_t^i - x_t^1)) + \sum_{k < t \leq r} (y_t^j \wedge z_t^i).$$

α_{ts}^j and y_t^j do not depend on i, for every $t \leq k$ there is some $s \leq k$ and some j such that $\alpha_{ts}^j \neq 0$ or $\alpha_{st}^j \neq 0$, and $\langle Y \rangle = \langle \{ y_t^j : 1 \leq t \leq r, 1 \leq j \leq r \} \rangle$.

(S2): For every permutation σ of $1, \ldots, h$ with $\sigma(1) = 1$ define $f_\sigma(y) = y$, $f_\sigma(x_t^i) = x_t^{\sigma(i)}$ for $1 \leq t \leq k$ and $f_\sigma(z_t^i) = z_t^{\sigma(i)}$ for $k < t \leq r$. Then f_σ induces an automorphism of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$ and the corresponding structure in $\mathcal{S}$.

(S3): Let $1 < j$. For $1 \leq t \leq k$ define $f_j(x_t^i) = x_t^i$ for $2 \leq i \leq h$ and $i \neq j$, $f_j(x_t^1) = x_t^j$, and $f_j(x_t^j) = x_t^1$. For $k < t \leq r$ define $f_j(z_t^i) = z_t^i - z_t^j$ for $2 \leq i \leq h$ and $i \neq j$, and $f_j(z_t^j) = -z_t^j$. Then f_j induces an automorphism of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$ and the corresponding structure in $\mathcal{S}$.

(S1) was already proved above. (S2) follows from Lemma 6.1. To obtain (S3) we repeat our considerations with X_j and X_1 exchanged, and compare the result with the situation in (S1) via Lemma 6.1.

Now we have to realize that there are many ways to choose a sequence of compatible bases $X_1, \ldots, X_h$ of $K_1, \ldots, K_h$ over H. Among all such, we choose one that gives us first a minimal k in (S1), and then a minimal $\langle Y \rangle$ in (S1). Let $X_i^+ - X_1^+$ be $\{ x_t^i - x_t^1 : 1 \leq t \leq k \}$.

We call a vector space automorphism f of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$ *well-tolerated* if

(a) $f(y) = y$ for $y \in Y$, $f(\langle Y X_i^+ \rangle) = \langle Y X_i^+ \rangle$, $f(\langle (X_i^+ - X_1^+) X_i^- \rangle) = \langle (X_i^+ - X_1^+) X_i^- \rangle$,

(b) for $1 \leq i, j \leq h$ the equality $f(x_t^i) = y + \sum_{1 \leq s \leq k} \gamma_s x_s^i$ holds if and only if

$$f(x_t^j) = y + \sum_{1 \leq s \leq k} \gamma_s x_s^j \quad \text{where } y \in Y,$$

(c) for $2 \leq i, j \leq r$ the equality $f(z_t^i) = \sum_{1 \leq s \leq k} \eta_s (x_s^i - x_s^1) + \sum_{k < s \leq r} \eta_s z_s^i$ holds if and only if

$$f(z_t^j) = \sum_{1 \leq s \leq k} \eta_s (x_s^j - x_s^1) + \sum_{k < s \leq r} \eta_s z_s^j.$$

Note that a well-tolerated automorphism f of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$ as above can be extended to a vector space automorphism g of $\langle Y X_1, \ldots, X_h \rangle$ with $g(\langle Y X_i \rangle) = \langle Y X_i \rangle$. By our choice of $Y, X_1, \ldots, X_h$ we obtain:

(S4): For every well-tolerated automorphism f of $\langle YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$ the sequence $Yf(X_1^+)f(X_2^+)f(X_2^-)\ldots f(X_h^+)f(X_h^-)$ again satisfies (S1) with the same k and $\langle Y \rangle$, (S2), and (S3).

Now let $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ be any sequence with $X_i^+ = \{x_1^i, \ldots, x_k^i\}$ and $X_i^- = \{z_{k+1}^i, \ldots, z_r^i\}$. We call $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ a *reduced k-sequence*, if $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ is linearly independent and $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ fulfills (S1), (S2), (S3), and (S4).

Let $(\Sigma 4)^m$ be the following property:

$(\Sigma 4)^m$**:** For every reduced k-sequence $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ in M with $k + |Y|, r < m$ we obtain

$$h \leq 2(max\{k + |Y|, r\}) + 1.$$

Furthermore we define:

$(\Sigma 4)$**:** For every reduced k-sequence $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ we obtain $h \leq 2(max\{k + |Y|, r\}) + 1$.

In the following lemma we summarize some trivial facts about reduced k-sequences:

Lemma 6.2. *Let $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ be a reduced k-sequence.*

1. $|Y| \leq r^2$.
2. *Linear combinations of the relations $\Phi_1^i, \ldots, \Phi_r^i$ in* (S1) *are again of the form described in* (S1).
3. *The composition of two well-tolerated automorphisms is again well-tolerated. Therefore we can iterate* (S4).
4. *Assume that M fulfills $(\Sigma 4)$. If $K_1, \ldots, K_h$ are realizations of a minimal extension of H that are in free composition over H, then $h \leq 2 \dim(K_1) + 1$.*

Proof. To prove (4) we construct a reduced k-sequence from H and $K_1, \ldots, K_h$ as described above. Then the assertion follows. $\qquad\square$

By Lemma 6.2.3 the application of well-tolerated automorphisms transforms a reduced k-system into a reduced k-system again. As above let A, B, and C be finite structures in $\mathcal{S}$ that fulfill $(\Sigma 3)$. Assume that A is a common substructure of B and C, and that A is selfsufficient in B and $(\dim(B/A) + 2 + l)$-selfsufficient in C for some l. Let D be $B \star_A C$. By Theorem 4.1 D fulfills $(\Sigma 3)^0$. Often we assume, furthermore, that B is a minimal extension of A that is not realized in C. Then D fulfills $(\Sigma 3)$.

Lemma 6.3. *Assume A, B, C, and $D = B \star_A C$ in $\mathcal{S}$ are chosen as described above such that D fulfills $(\Sigma 3)^0$. Let $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ be a reduced k-sequence in D with $0 < k$. Assume that Y is a suitable basis of $\langle Y \rangle$.*

1. *If $\langle YX_1^+ X_2^+ \rangle \cap C = \langle Y \rangle \cap C$, then $\langle YX_1^+ X_2^+ \rangle = \langle Y \rangle + (\langle YX_1^+ X_2^+ \rangle \cap B)$.*
2. *If $\langle YX_1^+ \rangle \cap C = \langle Y \rangle \cap C$, and $\langle YX_1^+ X_2^+ \rangle \cap B = \langle YX_1^+ \rangle \cap B$, then $\langle YX_1^+ \rangle = \langle Y \rangle + (\langle YX_1^+ \rangle \cap B)$, $\langle YX_2^+ \rangle = \langle Y \rangle + (\langle YX_2^+ \rangle \cap C)$, and $\langle YX_1^+ X_2^+ \rangle = \langle YX_1^+ \rangle + (\langle YX_1^+ X_2^+ \rangle \cap C)$.*
3. *If $\langle YX_1^+ X_2^+ \rangle \cap B = \langle Y \rangle \cap B$, then $\langle YX_1^+ X_2^+ \rangle = \langle Y \rangle + (\langle YX_1^+ X_2^+ \rangle \cap C)$.*
4. *If $\langle YX_1^+ \rangle \cap B = \langle Y \rangle \cap B$ and $\langle YX_1^+ X_2^+ \rangle \cap C = \langle YX_1^+ X_2^+ \rangle \cap C$, then $\langle YX_1^+ \rangle = \langle Y \rangle + (\langle YX_1^+ \rangle \cap C)$, $\langle YX_2^+ \rangle = \langle Y \rangle + (\langle YX_2^+ \rangle \cap B)$, and $\langle YX_1^+ X_2^+ \rangle = \langle YX_1^+ \rangle + (\langle YX_1^+ X_2^+ \rangle \cap B)$.*

Proof. Since we use bases in our proofs, we write the lemma in the language of suitable bases:

1. Assume that $YS_1^{+bx}S_1^{+e}$ is a suitable basis for $\langle YX_1^+ \rangle$. If $YS_2^{+bx}S_2^{+e}$ is a suitable basis for $\langle YX_2^+ \rangle$ and $YS_1^{+bx}S_1^{+e}Z_2^{+bx}Z_2^{+e}$ is a suitable basis for $\langle YX_1^+X_2^+ \rangle$, then $S_1^{+e} = S_2^{+e} = Z_2^{+e} = \emptyset$.

2. Assume that $YS_1^{+bx}S_1^{+e}$ is a suitable basis for $\langle YX_1^+ \rangle$. If $YS_2^{+cx}S_2^{+e}$ is a suitable basis for $\langle YX_2^+ \rangle$ and $YS_1^{+bx}S_1^{+e}Z_2^{+cx}Z_2^{+e}$ is a suitable basis for $\langle YX_1^+X_2^+ \rangle$, then $S_1^{+e} = S_2^{+e} = Z_2^{+e} = \emptyset$.

3. Assume that $YS_1^{+cx}S_1^{+e}$ is a suitable basis for $\langle YX_1^+ \rangle$. If $YS_2^{+cx}S_2^{+e}$ is a suitable basis for $\langle YX_2^+ \rangle$ and $YS_1^{+cx}S_1^{+e}Z_2^{+cx}Z_2^{+e}$ is a suitable basis for $\langle YX_1^+X_2^+ \rangle$, then $S_1^{+e} = S_2^{+e} = Z_2^{+e} = \emptyset$.

4. Assume that $YS_1^{+cx}S_1^{+e}$ is a suitable basis for $\langle YX_1^+ \rangle$. If $YS_2^{+bx}S_2^{+e}$ is a suitable basis for $\langle YX_2^+ \rangle$ and $YS_1^{+cx}S_1^{+e}Z_2^{+bx}Z_2^{+e}$ is a suitable basis for $\langle YX_1^+X_2^+ \rangle$, then $S_1^{+e} = S_2^{+e} = Z_2^{+e} = \emptyset$.

We show (1) and (2). The proofs of (3) and (4) are similar, since we do not use that A is selfsufficient in B and $(\dim(B/A) + 2 + l)$-selfsufficient in C.

We start with the proof of (1). First we assume that $Z_2^{+e} \neq \emptyset$ and show a contradiction. We rewrite (S1) with respect to $YS_1^{+bx}S_1^{+e}Z_2^{+bx}Z_2^{+e}$ to get a contradiction with the help of Lemma 4.3. First we apply several well-tolerated automorphisms of $\langle YX_1^+X_2^+X_2^- \ldots X_h^+X_h^- \rangle$ to make this easier.

By (S4) we can assume w.l.o.g. that

(1) $X_2^+ = S_2^{+bx}S_2^{+e}$.

Then there exists k_0 such that $1 \leq k_0 \leq k$ and $\{x_1^2, \ldots, x_{k_0}^2\} = S_2^{+bx}$. Using the notation of (S3) we have $x_i^1 = f_2^{-1}(x_i^2)$. Using (S4) for a suitable well-tolerated automorphism, we can assume that (1) remains true and w.l.o.g.:

(2) For every i with $1 \leq i \leq k_0$ either $x_i^1 + w \notin B$ for all $w \in \langle Y\{x_1^1, \ldots, x_{i-1}^1\}\rangle$ or there exists some $y_i \in \langle Y \rangle$ such that $x_i^1 - y_i \in B$.

In the first case we define $u_i^1 = x_i^1$, and in the second $u_i^1 = x_i^1 - y_i$. $Y\{u_1^1, \ldots, u_{k_0}^1\}$ can be extended to a suitable basis of $\langle YX_1^+ \rangle$. By application of (S4) to well-tolerated automorphisms of the form $f(x_i^j) = x_i^j - \sum_{l<i} r_l x_l^j - y$, where $y \in \langle Y \rangle$ and $k_0 < i$, and then of the form of a suitable permutation, we can assume w.l.o.g. that (1) and (2) remain true and:

(3) There exists k_1 with $k_0 \leq k_1 \leq k$ such that $\{u_i^1 : 1 \leq i \leq k_0, u_i \in B\} \cup \{x_{k_0+1}^1, \ldots, x_{k_1}^1\}$ can be considered as S_1^{+bx} and $\{u_i^1 : 1 \leq i \leq k_0, u_i \notin B\} \cup \{x_{k_1+1}^1, \ldots, x_k^1\}$ as S_1^{+e}.

For $k_0 \leq i \leq k$ we define $u_i^1 = x_i^1$. Then S_1^{+bx} can be chosen as the sequence of all u_i^1 with $u_i^1 \in B$ and S_1^{+e} as the sequence of all u_i^1 with $u_i^1 \notin B$. We have $X_2^+ = S_2^{+bx}S_2^{+e}$. But $X_1^+ \neq S_1^{+bx}S_1^{+e}$ is possible. To obtain equality as sets we have to add the y_i's, as defined above in (2). Now we want to extend $YS_1^{+bx}S_1^{+e}$ to a suitable basis of $\langle YX_1^+X_2^+ \rangle$. By assumption of the lemma $\langle YX_1^+X_2^+ \rangle$ has a suitable basis of the form $YS_1^{+bx}S_1^{+e}Z_2^{+bx}Z_2^{+e}$. Therefore

(4) $YS_1^{bx}S_1^{+e}\{x_1^2, \ldots, x_{k_0}^2\}$ is an initial part of a suitable basis of $\langle YX_1^+X_2^+ \rangle$.

We define $u_i^2 = x_i^2$ for $1 \leq i \leq k_0$. For $k_0 < i$ we define either

(a) $u_i^2 = x_i^2$, if x_i^2 is not in B modulo $\langle YS_1^{+e}\{u_j^2 : j < i\}\rangle$

or otherwise

(b) $u_i^2 = x_i^2 - w_i^+$, where $u_i^2 \in B \setminus A$ and $w_i^+ \in \langle Y^c Y^d S_1^{+e} \{u_l^2 : l < i, u_l^2 \notin B\}\rangle$.

Since by (3) $x_i^1 = u_i^1 \in S_1^{+bx}$ for $k_0 < i \le k_1$ and $x_i^1 = u_i^1 \in S_1^{+e}$ for $k_1 < i \le k$, we can apply (S4) again for well-tolerated automorphisms defined by $f(x_i^j) = x_i^j - \sum_{k_0 < l < i} r_l x_l^j$. Then (1), (2), (3), and (4) remain true, and

(5)   in case (b) $w_i^+ \in \langle Y^c Y^d S_1^{+e}\rangle$.

Now we can regard $\{u_i^j : j \in \{1,2\}, 1 \le i \le k\}$ as another enumeration of $S_1^{+bx} S_1^{+e} Z_2^{+bx} Z_2^{+e}$. We have the following connections with the basis $Y X_1^+ X_2^+$:
$u_i^1 = x_i^1 - y_i$ for $1 \le i \le k_0$, where $y_i \in \langle Y \rangle$, and $y_i \neq 0$ implies $u_i^1 \in S_1^{+bx}$
$u_i^2 = x_i^2 \in B$ for $1 \le i \le k_0$,
$u_i^1 = x_i^1$ for $k_0 < i \le k$, and
$u_i^2 = x_i^2 - w_i^+$ for $k_0 < i \le k$, where $w_i^+ = 0$ if $u_i^2 \notin B$ and $w_i^+ \in \langle Y^c Y^d S_1^{+e}\rangle$
otherwise.

First we show that $Z_2^{+e} = \emptyset$. That means that all u_i^2 are in B. Otherwise there is some $u_{i_0}^2 \notin B$. Then $k_0 < i_0$ and $u_{i_0}^2$ occurs in the unique representation of the x_j^l's ($l = 1, 2$) with respect to the basis $Y\{u_i^1 : 1 \le i \le k\}\{u_i^2 : 1 \le i \le k\}$ in the case $l = 2$ and $j = i_0$ only. By (S1) we know that $N(\langle Y X_1^+ X_2^+ X_2^- \rangle)$ is freely generated by some $\Phi_1^2, \ldots, \Phi_r^2$ over $N(\langle Y X_1^+ \rangle)$ that are of the following form:

$$\sum_{1 \le s < t \le k} \alpha_{st}^j \left((x_s^2 \wedge x_t^2) - (x_s^1 \wedge x_t^1) \right) + \sum_{1 \le t \le k} (y_t^j \wedge (x_t^2 - x_t^1)) + \sum_{k < t \le r} (y_t^j \wedge z_t^2).$$

For every t with $1 \le t \le k$ there exist s and j such that $s < t$ and $\alpha_{st}^j \neq 0$ or $t < s$ and $\alpha_{ts}^j \neq 0$. Choose l and j such that $l < i_0$ and $\alpha_{li_0}^j \neq 0$ or $i_0 < l$ and $\alpha_{i_0 l}^j \neq 0$. If Φ_j^2 is rewritten with respect to $Y\{u_i^1 : 1 \le i \le k\}\{u_i^2 : 1 \le i \le k\}$, then $u_l^2 \wedge u_{i_0}^2$ or $u_{i_0}^2 \wedge u_l^2$ respectively occurs exactly once in this representation of Φ_j^2. (5) is used. But this contradicts Lemma 4.3, if we consider a suitable basis of D that is consistent with $Y\{u_1^1, \ldots, u_k^1\}\{u_1^2, \ldots, u_k^2\}$.

It remains to show that $S_1^{+e} = \emptyset$. Then $S_1^{+e} = \emptyset$ and $Z_2^{+e} = \emptyset$ imply $S_2^{+e} = \emptyset$. Let us assume that $S_1^{+e} \neq \emptyset$. For $u_{i_0}^1 \in S_1^{+e}$ we have $u_{i_0}^1 = x_{i_0}^1$. If $u_{i_0}^1$ is not involved in the representation of any w_j^+ with respect to $Y^c Y^d S_1^{+e}$, then we obtain a contradiction as above:

We rewrite $\Phi_1^2, \ldots, \Phi_r^2$ in (S1) with respect to $Y\{u_1^1, \ldots, u_k^1\}\{u_1^2, \ldots, u_k^2\}$. Then there is some Φ_j^2 and some t such that Φ_j^2 contains exactly once the basic commutator $u_{i_0}^1 \wedge u_t^1$ or $u_t^1 \wedge u_{i_0}^1$ respectively. Again this is impossible by Lemma 4.3.

Therefore we assume now that $u_{i_0}^1$ occurs in some w_m^+. For all i with $k_0 < i \le k$ we define $w_i^+ = q_i u_{i_0} + w_i$, where $w_i \in \langle Y^c Y^d (S_1^{+e} \setminus \{u_{i_0}^1\})\rangle$. By assumption $q_m \neq 0$. Then $k_0 < m$. We want to procduce a situation such that $q_i \neq 0$ for exactly one i. Otherwise we fix m minimal with $q_m \neq 0$. We need $m \neq i_0$. If $i_0 \le k_0$, then this is true. If $k_0 < i_0$, then $k_1 < i_0$ by (3). In the case $k_1 < i_0 = m$ we choose another m_1 with $q_{m_1} \neq 0$. Then $m < m_1$. We can exchange x_m^j and $x_{m_1}^j$ by (S4). By construction (1), (2), (3), (4), and (5) remain true. Therefore we can assume w.l.o.g. that $m \neq i_0$ and m is minimal. Now we use (S4) again and apply the

following well-tolerated automorphism determined by

$$f(x_i^j) = \begin{cases} x_i^j - (q_i/q_m)x_m^j & \text{if } m < i \text{ and } x_i^2 = u_i^2 + q_i u_{i_0}^1 + w_i, \\ x_i^j & \text{otherwise.} \end{cases}$$

In the first case of the definition we have

$$f(x_i^2) = u_i^2 + q_i u_{i_0}^1 + w_i - (q_i/q_m)(u_m^2 + q_m u_{i_0}^1 + w_m)$$
$$= (u_i^2 - (q_i/q_m)u_m^2) + (w_i - (q_i/q_m)w_m)$$

and $f(x_i^1) = x_i^1 - (q_i/q_m)x_m^1$.

Note that $u_i^1 = x_i^1$ for $u_i^1 \in S_1^{+e}$; in particular, $u_{i_0}^1 = x_{i_0}^1$. Since $m \neq i_0$ we have for $k_0 \leq i \leq m$

$$w_i \in \langle Y^c Y^d f(S_1^{+e}) \setminus \{f(x_{i_0}^1)\}\rangle$$

and for $m < i$

$$w_i - (q_i/q_m)w_m \in \langle Y^e Y^d f(S_1^{+e} \setminus \{f(x_{i_0}^1)\})\rangle.$$

Hence after this application of (S4) we can assume that all our assumptions remain true and that $x_{i_0}^1 = u_{i_0}^1$ occurs in $w_m^+ = q_m u_{i_0}^1 + w_m$ where $q_m \neq 0$ but in no other w_i^+ with respect to the basis $Y^c Y^d S_1^{+e}$. Again we rewrite (S1) with respect to the basis

$$Y\{u_1^1, \ldots, u_k^1\}\{u_1^2, \ldots, u_k^2\}.$$

Then there is some Φ_j^2 in $N(\langle Y X_1^+ X_2^+ X_2^-\rangle)$ and some t such that $\alpha_{mt}^j \neq 0$ or $\alpha_{tm}^j \neq 0$, and $u_{i_0}^1 \wedge u_t^2$ or $u_t^2 \wedge u_{i_0}^1$ occurs in the representation of (S1) exactly once. Again we have a contradiction to Lemma 4.3.

To prove (2) we apply (S4) similarly as above and obtain:

(6) $X_2^+ = S_2^{+cx} S_2^{+e}$, which means $\{x_1^2, \ldots, x_{k_0}^2\} = S_2^{+cx}$ and $\{x_{k_0+1}^2, \ldots, x_k^2\} = S_2^{+e}$.

(7) For $1 \leq i \leq k_0$ either $x_i^1 = u_i^1 \in S_1^{+e}$, or there exists $y_i \in \langle Y\rangle$ with $u_i^1 = x_i^1 - y_i \in S_1^{+bx}$.

(8) There exists k_1 such that $k_0 \leq k_1 \leq k$,

$$\{u_i^1 : 1 \leq i \leq k_0, u_i^1 \in B\}\{x_{k_0+1}^1, \ldots, x_{k_1}^1\}$$

can be considered as S_1^{+bx}, and

$$\{u_i^1 : 1 \leq i \leq k_0, u_i^1 \notin B\}\{x_{k_1+1}^1, \ldots, x_k^1\}$$

can be considered as S_1^{+e}.

We define $u_i^1 = x_i^1$ for $k_0 < i \leq k$. Since $Y S_1^{+bx} S_1^{+e} Z_2^{+cx} Z_2^{+e}$ is a suitable basis for $\langle Y X_1^+ X_2^+\rangle$, we have:

(9) $Y S_1^{+bx} S_1^{+e} \{x_1^2, \ldots, x_{k_0}^2\}$ is the initial part of a suitable basis of $\langle Y X_1^+ X_2^+\rangle$.

We define $u_i^2 = x_i^2$ for $1 \leq i \leq k_0$. As in the proof of 1) we get w.l.o.g.:

(10) For $k_0 < i$
(a) either $u_i^2 = x_i^2$ is not in C modulo $\langle Y S_1^{+bx} S_1^{+e}\{u_j^2 : j < i\}\rangle$,
(b) or $u_i^2 = x_i^2 - w_i^+$ is in C for some $w_i^+ \in \langle Y^b Y^d S_1^{+bx} S_1^{+e}\rangle$.

In the first case u_i^2 is in Z_2^{+e} and in the second case u_i^2 is in Z^{+cx}. First we show that $Z_2^{+e} = \emptyset$. If $u_i^2 \in Z_2^{+e}$, then we know by (10) that it does not occur in the representation of any w_j^+. Therefore we obtain a contradiction as in the proof of 1. The next step is to show that $S_1^{+e} = \emptyset$. As above we assume that some $u_{i_0}^1 \in S_1^{+e}$. Then $u_{i_0}^1 = x_{i_0}^1$. If $u_{i_0}^1$ is not in the representation of some w_j^+, then we obtain a contradiction as above. Otherwise we show that we can assume w.l.o.g. that u_{i_0} occurs exactly once in some w_j^+ and get a contradiction.

It remains to show that $S_2^{+e} = \emptyset$. If $S_2^{+e} \neq \emptyset$, then by (6) and $S_1^{+e} = \emptyset$ we have $k_0 < k_1 = k$, and $x_i^2 \in S_2^{+e}$ for all $i > k_0$. Then $x_i^2 = u_i^2 + w_i^+$, where $u_i^2 \in C$ and $w_i^+ \in \langle Y^b Y^d S_1^{+bx} \rangle$ as in (10b) for $i > k_0$. w_i^+ has to contain some u_l^1 of S_1^{+bx}, since $Z_2^{+e} = \emptyset$. As in the proof of (1) we can show that w.l.o.g. some u_l^1 occurs in $w_{k_0+1}^+$ but not in any w_i^+ with $i > k_0 + 1$. Again we get a contradiction to Lemma 4.3, since some Φ_j^2 in (S1) has to contain a basic commutator $u_t^2 \wedge u_l^1$ or $u_l^1 \wedge u_t^2$ exactly once, where $u_t^2 \in Z_2^{cx}$ and $u_l^1 \in S_1^{+bx}$, a contradiction. $\square$

Now all our efforts are concentrated on the proof of the following:

Lemma 6.4. *Assume that A, B, and C are finite structures in $\mathcal{S}$ that satisfy $(\Sigma 3)$ and $(\Sigma 4)^m$. Let A be a common substructure of B and C, B a minimal extension of A that is not realized in C, and let A be n-selfsufficient in C, where $2m^2, \dim(B/A) + 2 \leq n$. Let D be $B \star_A C$. Then C is selfsufficient in D, B is n-selfsufficient in D, and D satisfies $(\Sigma 3)$ and $(\Sigma 4)^m$.*

By Theorem 4.1 it remains to show $(\Sigma 4)^m$ for D. Before we start we need three further lemmas.

Lemma 6.5. *Assume the situation of Lemma 6.4. Let $Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ be a reduced k-sequence with $r, |Y| + k < m$. For $2 \leq j \leq h$ we suppose that neither $\langle Y X_1^+ X_j^+ X_j^- \rangle \cap C = \langle Y X_1^+ \rangle \cap C$ nor $\langle Y X_1^+ X_j^+ X_j^- \rangle \cap B = \langle Y X_1^+ \rangle \cap B$ occurs. Then $h \leq 2(max\{|Y| + k, r\}) + 1$.*

Proof. Let $Q = Q^a Q^b Q^c Q^d$ be a suitable basis of $\langle Y X_1^+ \rangle$ that respects $\langle Y \rangle$. Assume that $S_j^a S_j^{by} S_j^{bx} S_j^{cx} S_j^e$ is a suitable extension of Q for $\langle Q X_j^+ X_j^- \rangle$. By the assumptions of the lemma at least one of the following conditions is fulfilled:

(a) $S_j^{bx} S_j^{cx} S_j^e = \emptyset$.

(b) $S_j^{bx} S_j^e \neq \emptyset$ and $S_j^a S_j^{by} S_j^{cx} \neq \emptyset$.

(c) $S_j^{cx} S_j^e \neq \emptyset$ and $S_j^a S_j^{by} S_j^{bx} \neq \emptyset$.

Let K be an extension of H. In the previous section, before Lemma 5.1, we described how to find a suitable basis of K, respecting a given suitable basis of H. $Q = Q^a Q^b Q^c Q^d$ is a suitable basis of $\langle Y X_1^+ \rangle$ that respects $\langle Y \rangle$. That means the following by definition:

$$Q^a = Y^a Z_1^a, \qquad Q^b = Y^b Z_1^{by} Z_1^{bx}, \qquad Q^c = Y^c Z_1^{cy} Z_1^{cx}, \qquad Q^d = Y^{e1} Z_1^e,$$

where $Y^a Y^b Y^c Y^d$ with $Y^d = Y^{e1}\{u + v \in \langle Y \rangle : u \in Z_1^{by}, v \in Z_1^{cy}\}$ is a suitable basis of $\langle Y \rangle$. This is the first step in an induction on $j \leq h$. For $1 \leq j \leq h$ we construct $Z_j = Z_j^a Z_j^{by} Z_j^{bx} Z_j^{cx} Z_j^e$ and $Z_j^{cy}, Y^{ej}, Z_1^{ej}, \ldots, Z_{j-1}^{ej}, Z_j^{ej} = Z_j^e$ such that:

- For every $u \in Z_j^{by}$ there is exactly one $v \in Z_j^{cy}$, and conversely for every $v \in Z_j^{cy}$ there is exactly one $u \in Z_j^{by}$, such that

$$u + v \in \langle Y X_1^+ X_2^+ X_2^- \ldots X_{j-1}^+ X_{j-1}^- \rangle.$$

- $\langle Y \rangle$ has the suitable basis

$$Y^a Y^b Y^c Y^{ej} \{u + v \in \langle Y \rangle : u \in Z_k^{by}, v \in Z_k^{cy}, 1 \le k \le j\},$$

where Y^a is a basis of $\langle Y \rangle \cap A$, $Y^a Y^b$ is a basis of $\langle Y \rangle \cap B$, $Y^a Y^c$ is a basis of $\langle Y \rangle \cap C$, and

$$Y^{ej} \{u + v \in \langle Y \rangle : u \in Z_k^{by}, v \in Z_k^{bc}, 1 \le k \le j\}$$

is the part that is linearly independent over $(\langle Y \rangle \cap B) + (\langle Y \rangle \cap C)$.

- $\langle Y X_1^+ \rangle$ has the suitable basis

$$Y^a Z_1^a \ Y^b Z_1^{by} Z_1^{bx} \ Y^c Z_1^{cy} Z_1^{cx} \ Y^{ej} Z_1^{ej} \{u + v \in \langle Y X_1^+ \rangle :$$
$$u \in Z_k^{by}, v \in Z_k^{cy}, 2 \le k \le j\},$$

where $Y^a Z_1^a$ is a basis of $\langle Y X_1^+ \rangle \cap A$, $Y^a Z_1^a Y^b Z_1^{by} Z_1^{bx}$ is a basis of $\langle Y X_1^+ \rangle \cap B$, $Y^a Z_1^a Y^c Z_1^{cy} Z_1^{cx}$ is a basis of $\langle Y X_1^+ \rangle \cap C$, and

$$Y^{ej} Z_1^{ej} \{u + v \in \langle Y X_1^+ \rangle : u \in Z_k^{by}, v \in Z_k^{cy}, 2 \le k \le j\}$$

is a basis of $\langle Y X_1^+ \rangle$ over $(\langle Y X_1^+ \rangle \cap B) + (\langle Y X_1^+ \rangle \cap C)$.

- For $\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle$ with $1 < i \le j$ we obtain a basis

$$Y^a Z_1^a Z_2^a \ldots Z_i^a, Y^b Z_1^{by} Z_1^{bx} Z_2^{by} Z_2^{bx} \ldots Z_i^{by} Z_i^{bx}, Y^c Z_1^{cy} Z_1^{cx} Z_2^{cy} Z_2^{cx} \ldots Z_i^{cy} Z_i^{cx},$$

$$Y^{ej} Z_1^{ej} Z_2^{ej} \ldots Z_i^{ej} \{u + v \in \langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle :$$
$$u \in Z_k^{by}, v \in Z_k^{cy}, i < k \le j\},$$

where the a-part is a basis for

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle \cap A,$$

the union of the a-part and the b-part is a basis for

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle \cap B,$$

the union of the a-part and the c-part is a basis for

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle \cap C,$$

and the remaining part is a basis of

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle$$

over $(\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle \cap B) + (\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle \cap C)$.

The Z_j are constructed by induction on j. Assume Z_j has been built; we start the construction of Z_{j+1}. Z_{j+1}^a is obtained as usual. Then $Z_{j+1}^{by} = \bigcup_{0 \le i \le j} Z_{j+1}^{by}(i)$ and $Z_{j+1}^{cy} = \bigcup_{0 \le i \le j} Z_{j+1}^{cy}(i)$, where $Z_{j+1}^{by}(i)$ and $Z_{j+1}^{cy}(i)$ are obtained by induction on i. Let $Z_{j+1}^{by}(0)$ be a basis of $(\langle Y \rangle + C) \cap B \cap \langle Y X_1^+ X_2^+ X_2^- \ldots X_{j+1}^+ X_{j+1}^- \rangle$ over $(B \cap \langle Y X_1^+ X_2^+ X_2^- \ldots X_j^+ X_j^- \rangle) + \langle Z_{j+1}^a \rangle$. To get $Z_{j+1}^{cy}(0)$, choose for every $u \in Z_{j+1}^{by}(0)$ some v in C such that $u + v \in \langle Y \rangle$. If $Z_{j+1}^{by}(i)$ and $Z_{j+1}^{cy}(i)$ are constructed, then define $Z_{j+1}^{by}(i+1)$ to be a basis of $(\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i+1}^+ X_{i+1}^- \rangle + C) \cap B \cap \langle Y X_1^+ X_2^+ X_2^- \ldots X_{j+1}^+ X_{j+1}^- \rangle$ over

$$(B \cap \langle Y X_1^+ X_2^+ X_2^- \ldots X_j^+ X_j^- \rangle) + \langle Z_{j+1}^a Z_{j+1}^{by}(0) \ldots Z_{j+1}^{by}(i) \rangle$$

To obtain $Z_{j+1}^{cy}(i+1)$ we choose for every $u \in Z_{j+1}^{by}(i+1)$ some $v \in C$ such that $u + v \in \langle Y X_1^+ X_2^+ X_2^- \ldots X_{i+1}^+ X_{i+1}^- \rangle$. Then Z_{j+1}^{bx} and Z_{j+1}^{cx} are obtained as usual. $Y^{e(j+1)}$ is a basis of $\langle Y \rangle$ over

$$(\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j+1}^+ X_{j+1}^- \rangle \cap B) + (\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j+1}^+ X_{j+1}^- \rangle \cap C)$$

and, for $1 \le i \le j + 1$, $Z_i^{e(j+1)}$ is a basis of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle$ over

$$(\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j+1}^+ X_{j+1}^- \rangle \cap B)$$
$$+ (\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j+1}^+ X_{j+1}^- \rangle \cap C) + \langle Y_{j+1}^e Z_{j+1}^{e1} \ldots Z_{i-1}^{e(j+1)} \rangle.$$

Now after modifying Y^d and all Z_j^e we can assume:

$$Y^d = Y^{eh} \{ u + v \in \langle Y \rangle : u \in Z_l^{by}, v \in Z_l^{cy} \quad \text{for some } l \text{ with } 1 \le l \le h \}$$

and

$$Z_j^e = Z_j^{eh} \{ u + v \in \langle Z_j^e \rangle : u \in Z_l^{by}, v \in Z_l^{cy} \quad \text{for some } l \text{ with } \quad j < l \le h \}$$

for $1 \le j \le h$. We say:

$$Y Z_1 \ldots Z_h \text{ gives us a suitable basis of } \langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$$
$$\text{that respects } \langle Y \rangle, \langle Y X_1^+ \rangle, \ldots, \langle Y X_1^+ X_2^+ X_2^- \ldots X_{h-1}^+ X_{h-1}^- \rangle.$$

If $Z_j = Z_j^a Z_j^{by}$, then we can carry out the procedure above so that we have $Z_j^+ = Z_j^{+a} Z_j^{+by}$ as a suitable extension for

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j-1}^+ X_{j-1}^- X_j^+ \rangle$$

over $\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j-1}^+ X_{j-1}^- \rangle$ and $Z_j^- = Z_j^{-a} Z^{-by}$ as a suitable extension for

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_j^+ X_j^- \rangle$$

over $\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j-1}^+ X_{j-1}^- X_j^+ \rangle$.

Note that $Q^d = Y^d Z_1^e$. We suppose that $Q^d = \{ u_1 + v_1, \ldots, u_{l_1} + v_{l_1} \}$. By our assumption either $u_i + v_i$ is not in

$$(B \cap \langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle) + (C \cap \langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle)$$

or $u_i \in Z_j^{by}$ and $v_i \in Z_j^{cy}$ for some j with $2 \le j \le h$. Define $U_1 = Q^b \{ u_1, \ldots, u_{l_1} \}$, $V_1 = Q^c \{ v_1, \ldots, v_{l_1} \}$, and $W_1 = Q^a$.

If U_{j-1}, V_{j-1}, and W_{j-1} are defined and $Z_j = Z_j^a Z_j^{by} Z_j^{bx} Z_j^{cx} Z_j^e$, where $Z_j^e = \{ u_{l_{j-1}+1} + v_{l_{j-1}+1}, \ldots, u_{l_j} + v_{l_j} \}$, then we define

$U_j = U_{j-1} Z^{bx} \{ u_{l_{j-1}+1}, \ldots, u_{l_j} \}$,
$V_j = V_{j-1} Z^{cx} \{ v_{l_{j-1}+1}, \ldots, v_{l_j} \}$, and
$W_j = W_{j-1} Z_j^a$.

By assumption $u_i + v_i \in Z_j^e$ is either not in

$$(B \cap \langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle) + (C \cap \langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle)$$

or $u_i \in Z_l^{by}$ and $v_i \in Z_l^{cy}$ for some l with $j < l \le h$.

Now the fact that (a), (b), or (c) is true implies that one of the following conditions is fulfilled:

I. $Z_j^{bx} Z_j^{cx} Z_j^e = \emptyset$.

II. $Z_j^{bx} Z_j^e \ne \emptyset$ and $Z_j^a Z_j^{by} Z_j^{cx} \ne \emptyset$.

III. $Z_j^{cx} Z_j^e \ne \emptyset$ and $Z_j^a Z_j^{by} Z_j^{bx} \ne \emptyset$.

First we assume that $Y = Y^a \subseteq A$ and there is some $X_j^+ \subseteq A$. By (S3) w.l.o.g. $X_1^+ \subseteq A$. The case $X_1^+ = \emptyset$ ($k = 0$) is included. Then $Q = Q^a \subseteq A$. By (S1) a suitable basis of $\langle Y X_1^+ X_j^+ X_j^- \rangle$ cannot be of the form $Q S_j^e$ for any j. If $k = 0$, use Lemma 4.4. Hence $(\langle Y X_1^+ X_j^+ X_j^- \rangle \cap B) + (\langle Y X_1^+ X_j^+ X_j^- \rangle \cap C) \neq \langle Y X_1^+ \rangle$ for all j. Then Lemma 5.2 implies that either all $X_j^+ X_j^-$ are in B or they are all in C. $(\Sigma 4)^m$ for B and C gives the assertion. In all other cases we prove the following claim (11) that implies the assertion, as we will explain.

(11) There are functions f and g with $f(1) = g(1) = 0$ and $f(i-1) \leq f(i)$, $g(i-1) \leq g(i)$ for $2 \leq i \leq h$ and relations $\Delta_1, \ldots, \Delta_{f(j)} \in N(B)$ and $\Theta_1, \ldots, \Theta_{g(j)} \in N(C)$ such that for $2 \leq j \leq h$:
   (i) If we have Case II, then $f(j) = f(j-1) + |U_j \setminus U_{j-1}| + 1$.
   (ii) If we have Case III, then $g(j) = g(j-1) + |V_j \setminus V_{j-1}| + 1$.
   (iii) If we have Case I, then $f(j-1) \leq f(j) \leq f(j-1) + 1$, $g(j-1) \leq g(j) \leq g(j-1) + 1$, and $f(j) = f(j-1) + 1$ or $g(j) = g(j-1) + 1$.
   (iv) $\Delta_1, \ldots, \Delta_{f(j)}$ are linearly independent over $\wedge^2 A$.
   (v) $\Theta_1, \ldots, \Theta_{g(j)}$ are linearly independent over $\wedge^2 A$.
   (vi) If $f(j-1) < i \leq f(j)$, then Δ_i is in $(\wedge^2 \langle U_j W_j \rangle + \wedge^2 A) \cap N(B)$. If Δ_i is written as a linear combination of basic commutators with respect to a suitable basis of D that extends $U_j W_j$, then every u_l that is not in $\bigcup_{2 \leq s \leq j} Z_s^{by}$ occurs in basic commutators of the form $a \wedge u$ only, where $a \in A$.
   (vii) If $g(j-1) < i \leq g(j)$, then Θ_i is in $(\wedge^2 \langle V_j W_j \rangle + \wedge^2 A) \cap N(C)$. If Θ_i is written as a linear combination of basic commutators with respect to a suitable basis of D, that extends $V_j W_j$, then every v_l, that is not in $\cup_{2 \leq s \leq j} Z_s^{cy}$, occurs in basic commutators of the form $a \wedge v$ only, where $a \in A$.

First we show that (11.i-vii) implies $h \leq 2(k + |Y|) + 1$. Otherwise w.l.o.g. $h = 2(k + |Y|) + 2$. By (11.i-iii) we have

$$(f(h) - |U_h \setminus U_1|) + (g(h) - |V_h \setminus V_1|) \geq h - 1 > 2(k + |Y|).$$

It follows that

$$f(h) > k + |Y| + |U_h| - |U_1| \geq |U_h| \quad \text{or} \quad g(h) > k + |Y| + |V_h| - |V_1| \geq |V_h|.$$

By (11.iv) or (11.v) respectively we have

$$\dim(N(\langle U_h A \rangle)) - \dim(N(A)) > |U_h|, \quad \text{or} \quad \dim(N(\langle V_h A \rangle)) - \dim(N(A)) > |V_h|.$$

This is equivalent to

$$\delta(\langle U_h A \rangle) < \delta(A) \quad \text{or} \quad \delta(\langle V_h A \rangle) < \delta(A).$$

The first inequality contradicts the self-sufficiency of A in B. Note that

$$|V_h| \leq |Y| + |X_1^+| + (h-1)|X_2^+ X_2^-| < 2m^2,$$

because $k + |Y|, r < m$. Therefore the second inequality contradicts the $2m^2$-selfsufficiency of A in C.

Now we assume that $Y \not\subseteq A$ or $Y \subseteq A$ but no X_j^+ is in A. To prove (11) by induction on j we suppose that (11) is true for $j - 1$.

First we show (11) for j in the Cases II and III. Let us assume Case II; Case III is similar. We have $Z_j^{bx} Z_j^e \neq \emptyset$ and $Z_j^a Z_j^{by} Z_j^{cx} \neq \emptyset$. Then $f(j) = f(j-1) + |Z_j^{bx} Z_j^e| + 1$.

Since $\langle QZ_2 \dots Z_j \rangle$ is a minimal extension of $\langle QZ_2 \dots Z_{j-1} \rangle$, we obtain

$$\delta(\langle QZ_2 \dots Z_{j-1} Z_j^a Z_j^{by} Z_j^{cx} \rangle) > \delta(\langle QZ_2 \dots Z_{j-1} \rangle) = \delta(\langle QZ_2 \dots Z_j \rangle).$$

Therefore

$$\dim(N(\langle QZ_2 \dots Z_j \rangle)) - \dim(N(\langle QZ_2 \dots Z_{j-1} Z_j^a Z_j^{by} Z_j^{cx} \rangle)) > |Z_j^{bx} Z_j^e|$$

Denote $|Z_j^{bx} Z_j^e| + 1 = f(j) - f(j-1)$ by s. We choose relations $\Phi_1, \dots, \Phi_s$ in $N(\langle QZ_2 \dots Z_j \rangle)$ that are linearly independent over

$$N(\langle QZ_2 \dots Z_{j-1} Z_j^a Z_j^{by} Z_j^{cx} \rangle).$$

This is equivalent to the following:

> Every non-trivial linear combination of $\Phi_1, \dots \Phi_s$ considered as
> a linear combination of basic commutators over $QZ_2 \dots Z_j$
> contains a basic commutator with an element of $Z_j^{bx} Z_j^e$.

If we rewrite Φ_i as a linear combination of basic commutators with respect to $U_j V_j W_j$ then by Lemma 4.3 $\Phi_i = \Phi_i^b + \Phi_i^c$, where $\Phi_i^b \in \wedge^2 \langle W_j U_j \rangle$, $\Phi_i^c \in \wedge^2 \langle W_j V_j \rangle$, and there is some $J_i \in \wedge^2 A$ such that $\Phi_i^b + J_i \in N(B)$ and $\Phi_i^c - J_i \in N(C)$. We define $\Delta_{f(j-1)+i} = \Phi_i^b + J_i$. Then $\Delta_{f(j-1)+i} \in (\wedge^2 \langle W_j U_j \rangle + \wedge^2 A) \cap N(B)$. By construction every non-trivial combination of $\Delta_{f(j-1)+1}, \dots, \Delta_{f(j)}$, represented as a linear combination of basic commutators over a suitable basis of D that extends $W_j U_j$, contains a basic commutator with an element of $U_j \setminus U_{j-1}$. Therefore $\Delta_1, \dots, \Delta_{f(j)}$ are linearly independent over $\wedge^2 A$ by induction. It remains to show (11) (vi) for $\Delta_{f(j-1)+1}, \dots, \Delta_{f(j)}$. This is an immediate consequence of Lemma 4.3, if we use the suitable basis $QZ_2 \dots Z_j$.

Now let us assume that $Y^b Y^d \neq \emptyset$. We show (11) for j in the remaining Case I. We have $Z_j = Z_j^a Z_j^{by}$. Take $y \in Y^b Y^d$. Define $y_b = y$, if $y \in Y^b$, and $y_b \in U_1$, $y_c \in V_1$, if $y = y_b + y_c \in Y^d$. By (S1) $N(\langle Y X_1^+ X_j^+ X_j^- \rangle)$ is freely generated by $\Phi_1^j \dots \Phi_r^j$, where Φ_i^j is of the form

$$\sum_{1 \le s < t \le k} \alpha_{st}^i \left((x_s^j \wedge x_t^j) - (x_s^1 \wedge x_t^1) \right) + \sum_{1 \le t \le k} (y_t^i \wedge (x_t^j - x_t^1)) + \sum_{k < t \le r} (y_t^i \wedge z_t^j).$$

If we rewrite the y_t^i's with respect to Y, then let γ_t^i be the coefficient of $y \wedge (x_t^j - x_t^1)$ and $y \wedge z_t^j$ respectively in Φ_i^j.

As mentioned above we have chosen $Z_j^+ = Z_j^{+a} Z_j^{+by}$ as a suitable extension for $\langle Y X_1^+ X_2^+ X_2^- \dots X_{j-1}^+ X_{j-1}^- X_j^+ \rangle$ over $\langle Y X_1^+ X_2^+ X_2^- \dots X_{j-1}^+ X_{j-1}^- \rangle$ and $Z_j^- = Z_j^{-a} Z_j^{-by}$ for $\langle Y X_1^+ X_2^+ X_2^- \dots X_j^+ X_j^- \rangle$ over $\langle Y X_1^+ X_2^+ X_2^- \dots X_{j-1}^+ X_{j-1}^- X_j^+ \rangle$. Using an appropriate well-tolerated automorphism of $\langle Y X_1^+ X_2^+ X_2^- \dots X_j^+ X_j^- \rangle$, we may assume by (S4) that

$$\{x_1^j, \dots, x_k^j\} = Z_j^+ \text{ modulo } \langle Y X_1^+ X_2^+ X_2^- \dots X_{j-1}^+ X_{j-1}^- \rangle,$$

$$\{z_{k+1}^j, \dots, z_r^j\} = Z_j^- \text{ modulo } \langle Y X_1^+ X_2^+ X_2^- \dots X_{j-1}^+ X_{j-1}^- \rangle.$$

Let x_t be the element of Z_j^+ that corresponds to x_t^j, and let z_t be the element of Z_j^- that corresponds to z_t^j. In the following we will find suitable $x_t = a \in Z_j^{+a}$ or $z_t = a \in Z_j^{-a}$ or $x_t = u \in Z^{+by}$ or $z_t = u \in Z_j^{-by}$ such that $y_b \wedge a$ or $y_c \wedge a$ or $y_b \wedge u$

or $y_c \wedge v$ has a non-zero coefficient in the representation of some Φ_i^j with respect to $W_j U_j V_j$, where $v \in Z_j^{cy}$ is obtained by $u + v \in \langle Y X_1^+ X_2^+ X_2^- \ldots X_{j-1}^+ X_{j-1}^- \rangle$.

First we assume that we can choose $y = y_b \in Y^b$ or $y \in Y^d$ with $y = y_b + y_c$ where $y_b \in Z_1^{by} Z_2^{by} \ldots Z_{j-1}^{by}$. Then $x_t^j = x_t + x_t^0 + \epsilon_t y_b$ and $z_t^j = z_t + z_t^0$, where $x_t^0 + \epsilon_t y_b$ and z_t^0 are in $\langle Q Z_2 \ldots Z_{j-1} \rangle$ and $x_t^0 \in \langle Q Z_2 \ldots Z_{j-1} \setminus \{y_b\} \rangle$. In the representation of Φ_i^j with respect to $Q Z_2 \ldots Z_j$ the coefficient η_i^i of $y_b \wedge z_t$ is γ_t^i and of $y_b \wedge x_t$ is $\gamma_t^i + \sum_{s < t} \alpha_{st}^i \epsilon_s - \sum_{t < s} \alpha_{ts}^i \epsilon_s$. There has to be some i such that at least one of these coefficients η_t^i is not zero. Otherwise the well-tolerated automorphism e defined by $e(x_t^j) = x_t^j - \epsilon_t y$ and $e(z_t^j) = z_t^j$ would give an (S1)-representation of the Φ_i^j's with respect to $(Y \setminus \{y\}) X_1^+ X_j^+ X_j^-$, a contradiction to (S4). Fix i and t with $\eta_t^i \neq 0$. If $x_t = a \in Z_j^{+a}$ for $t \leq k$ or $z_t = a \in Z_j^{-a}$ for $k < t$, then Φ_i^j contains $y_b \wedge a$ with the coefficient η_t^i in its representation with respect to the basis $W_j U_j V_j$, because $a \notin W_{j-1} U_{j-1} V_{j-1}$. Otherwise $x_t = u \in Z_j^{+by}$ for $t \leq k$ or $z_t = u \in Z_j^{-by}$ for $k < t$. Note that $u \neq y_b$. Then there is some $v \in Z_j^{+cy}$ or $v \in Z_j^{-cy}$ such that $u + v \in Q^d Z_2^e \ldots Z_{j-1}^e$. In the representation of Φ_i^j with respect to $W_j U_j V_j$ the coefficient of $y_b \wedge u$ is $(\eta_t^i + \mu)$, where μ is the coefficient of $y_b \wedge v$ in this representation. Either $\eta_t^i + \mu \neq 0$ or $\mu \neq 0$. In the first case $y_b \wedge u$ occurs in this representation with the coefficient $(\eta_t^i + \mu) \neq 0$. In the second case $y_b \wedge v$ has the coefficient $\mu \neq 0$. This is impossible by Lemma 4.3.

Now we suppose that $Y^b = \emptyset$ and there are no $l < j$, $y_b \in Z_l^{by}$, and $y_c \in Z_l^{cy}$ such that $y_b + y_c \in Y^d$. By the construction of $Z_1, \ldots, Z_h$ we have that Y^d is linearly independent over $(\langle Q Z_1 \ldots Z_{j-1} \rangle \cap B) + (\langle Q Z_1 \ldots Z_{j-1} \rangle) \cap C)$. By assumption $Y^d \neq \emptyset$. As above, consider $x_t^j = x_t + x_t^0 + \epsilon_t y$, $z_t^j = z_t + z_t^0$, where $x_t^0 + \epsilon_t y, z_t^0 \in \langle Q Z_2 \ldots Z_{j-1} \rangle$ and $x_t^0 \in \langle Q Z_2 \ldots Z_{j-1} \setminus \{y\} \rangle$. Again $\eta_t^i = \gamma_t^i$ is the coefficient of $y \wedge z_t$ and $\eta_t^i = \gamma_t^i + \sum_{s < t} \alpha_{st}^i \epsilon_s - \sum_{t < s} \alpha_{ts}^i \epsilon_s$ is the coefficient of $y \wedge x_t$ for $t \leq k$ respectively in the representation of Φ_i^j with respect to the basis $Q Z_2 \ldots Z_j$. By the same argument as above we obtain i and t with $\eta_t^i \neq 0$. We have to discuss the following cases:

(i) Y^d is linearly independent over $(\langle Q Z_1 \ldots Z_j \rangle \cap B) + (\langle Q Z_1 \ldots Z_j \rangle \cap C)$.

(ii) There are some $y_b \in Z_j^{by}$ and $y_c \in Z_j^{cy}$ such that $y = y_b + y_c \in Y^d$.

Again we have used the fact that $Y Z_1 \ldots Z_h$ gives us a suitable basis of

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$$

respecting

$$\langle Y \rangle, \langle Y X_1^+ \rangle, \ldots, \langle Y X_1^+ X_2^+ X_2^- \ldots X_{h-1}^+ X_{h-1}^- \rangle.$$

We consider Case (i) and choose any $y \in Y^d$. Then there is no $y_b \in Z_j^{by}$ with $y - y_b \in Z_j^{cy}$. Let μ_t^i be the coefficient of $y \wedge z_t$ or $y \wedge x_t$ respectively in the representation of Φ_i^j with respect to the basis $W_j U_j V_j$. If $x_t = a \in Z_j^{+a}$ or $z_t = a \in Z_j^{-a}$ respectively, then $\mu_t^i = \eta_t^i \neq 0$ as above. Otherwise $x_t = u \in Z_j^{+by}$ or $z_t = u \in Z_j^{-by}$ respectively. Choose $v \in Z_j^{cy}$ such that $u + v \in Q^d Z_2^e \ldots Z_{j-1}^e$. By assumption $(u + v) \neq y$. Hence $\mu_t^i = \eta_t^i + \mu$, where μ is the coefficient of $y \wedge (u + v)$ in the representation of Φ_i^j with respect to the basis $Q Z_2 \ldots Z_j$. If we consider the representation of Φ_i^j with respect to the basis $W_j U_j V_j$, then we have $\mu = 0$ by Lemma 4.3. This case is not possible, since $\eta_t^i \neq 0$ would imply a contradiction.

There remains Case (ii). We choose $y \in Y^d$ with some $y_b \in Z_j^{by}$ such that $y - y_b \in Z_j^{cy}$. Assume $y_b = x_{t_0}$ or $y_b = z_{t_0}$ respectively. Then $\eta_{t_0}^i = 0$ for all i by Lemma 4.3. Note i and t are fixed above with $\eta_t^i \neq 0$. Therefore x_t or z_t respectively is different from y_b. If $x_t = a \in Z_j^{+a}$ or $z_t = a \in Z_j^{-a}$ respectively, then $\eta_t^i \neq 0$ is the coefficient of $y_c \wedge a$ in the representation of Φ_i^j with respect to the basis $W_j U_j V_j$. Otherwise $x_t = u \in Z_j^{+by}$ or $z_t = u \in Z_j^{-by}$ respectively. Choose $v \in Z_j^{cy}$ such that $u + v \in Q^d Z_2^e \ldots Z_{j-1}^e$. Let μ be the coefficient of $y \wedge (u + v)$ in the presentation of Φ_i^j with respect to $Q Z_2 \ldots Z_j$. Define $\nu_t^i = 0$ for $k < t$, $\nu_t^i = \alpha_{t_0 t}^i$ for $t_0 < t \leq k$, and $\nu_t^i = -\alpha_{t t_0}^i$ if $t < t_0 \leq k$. We consider again the representation of Φ_i^j with respect to the basis $W_j U_j V_j$. Then

$$\eta_t^i + \nu_t^i + \mu \text{ is the coefficient of } y_b \wedge u;$$

$$\eta_t^i + \mu \text{ is the coefficient of } y_c \wedge u;$$

$$\mu \text{ is the coefficient of } y_c \wedge v.$$

By Lemma 4.3, $\eta_t^i + \mu = 0$. If $y_b \wedge u$ has the coefficient 0, then $\nu_t^i = 0$ and $\eta_t^i \neq 0$ implies $\mu \neq 0$. Hence either $y_b \wedge u$ or $y_c \wedge v$ has a non-zero coefficient.

Our assumption was that $Y^b Y^d \neq 0$. In all possible cases we have shown that there is some Φ_i^j that contains a basic commutator $y_b \wedge a$ or $y_c \wedge a$ or $y_b \wedge u$ or $y_c \wedge v$ in its representation with respect to $W_j U_j V_j$, where $a \in Z_j^a$, $u \in Z_j^{by}$, and $v \in Z_j^{cy}$. Of course $y_b \wedge a$ cannot occur in the representation of any Δ_i with $i < f(j)$ as a linear combination of basic commutators with respect to $W_j U_j V_j$. The same is true for $y_b \wedge u$ and Δ_i with $i < f(j)$ and for $y_c \wedge v$ and Θ_i with $i < g(j)$, since the representation as a linear combination of basic commutators with respect to $W_j U_j V_j$ of Δ_i with $i < f(j)$ and of Θ_i with $i < g(j)$ contains $w \wedge u$ or $w \wedge v$ respectively for $w \in A$ only, by (11) (vi) and (vii). If we represent Φ_i^j as a linear combination of basic commutators with respect to a suitable basis of D that contains $W_j U_j V_j$, then $\Phi_i^j = \Phi^b + \Phi^c$ and there is some $J \in \wedge^2 A$ such that $\Phi^b \in \wedge^2 \langle W_j U_j \rangle$, $\Phi^c \in \wedge^2 \langle W_j V_j \rangle$, $\Phi^b + J \in N(B)$, and $\Phi^c - J \in N(C)$. We define $f(j) = f(j-1) + 1$ and $\Delta_{f(j)} = \Phi^b + J$, if $y_b \wedge a$ or $y_b \wedge u$ occurs. Since $y_b \wedge a$ or $y_b \wedge u$ occurs in Δ_l for $1 \leq l \leq f(j)$ if and only if $l = f(j)$, we obtain the linear independence of $\Delta_1, \ldots, \Delta_{f(j)}$ by induction. Analogously we define $g(j) = g(j-1) + 1$ and $\Theta_{g(j)} = \Phi^c - J$, if $y_c \wedge a$ or $y_c \wedge v$ occurs. The linear independence of $\Theta_1, \ldots, \Theta_{g(j)}$ follows. Again (11) (vi) and (vii) are obtained by Lemma 4.3. For the case $Y^b Y^d \neq \emptyset$ (11), and therefore the lemma are proved.

If $Y^b Y^d = \emptyset$ but $Y^c \neq \emptyset$, we prove Case I of (11) in the same way. We define $\Delta_{f(j)}$ and $\Theta_{g(j)}$ analogously as above, working with Z_j^{cy} instead of Z_j^{by}.

Now we carry out the induction assuming that $Y = Y^a \subseteq A$. By the assumption above $X_j^+ \nsubseteq A$ for all j. Then $k > 0$. Assume (11) is shown for $j - 1$. It remains to show (11) for j in Case I. As mentioned above, Z_j is split into $Z_j^+ = Z_j^{+a} Z_j^{+by}$ and $Z_j^- = Z_j^{-a} Z_j^{-by}$ such that $Y Z_1 \ldots Z_j^+ Z_j^- \ldots Z_h$ gives us a suitable basis of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$ that respects

$$\langle Y \rangle, \langle Y X_1^+ \rangle, \langle Y X_1^+ X_2^+ X_2^- \rangle, \ldots, \langle Y X_1^+ X_2^+ X_2^- \ldots X_{j-1}^+ X_{j-1}^- \rangle,$$

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j-1}^+ X_{j-1}^- X_j^+ \rangle, \langle Y X_1^+ X_2^+ X_2^- \ldots X_j^+ X_j^- \rangle,$$

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j+1}^+ X_{j+1}^- \rangle, \ldots, \langle Y X_1^+ X_2^+ X_2^- \ldots X_{h-1}^+ X_{h-1}^- \rangle.$$

The image of X_j^+ and the image of Z_j^+ form bases of the vector space

$$\langle YX_1^+X_2^+X_2^-\ldots X_{j-1}^+X_{j-1}^-X_j^+\rangle$$

over $\langle YX_1^+X_2^+X_2^-\ldots X_{j-1}^+X_{j-1}^-\rangle$. If $Z_j^{+a}Z_j^{+by}=\{t_1^j,\ldots,t_k^j\}$, then by (S4) we can assume w.l.o.g. that $x_i^j=t_i^j+q_i^j$, where $q_i^j\in\langle YX_1^+X_2^+X_2^-\ldots X_{j-1}^+X_{j-1}^-\rangle$. If $t_i^j\in Z_j^{+a}$, then t_i^j does not occur in $W_{j-1}U_{j-1}V_{j-1}$. If $t_i^j\in Z_j^{+by}$, then $t_i^j\in U_{j-1}$ and there exists some $w_i^j\in V_{j-1}$ such that $t_i^j+w_i^j\in Q^dZ_2^e\ldots Z_{j-1}^e$ but t_i^j and w_i^j are not in $\langle YX_1^+X_2^+X_2^-\ldots X_{j-1}^+X_{j-1}^-\rangle$. If we rewrite q_i^j with respect to $W_{j-1}U_{j-1}V_{j-1}$, then we have $x_i^j=(1+\gamma_i)t_i^j+\gamma_iw_i^j+q_i$ where $q_i\in\langle W_{j-1}U_{j-1}V_{j-1}\setminus\{t_i^j,w_i^j\}\rangle$. If $t_i^j\in Z_j^{+a}$, then we use the same representation with $\gamma_i=0$ and $w_i^j=0$. For $1\leq i\leq k$ we define $x_i^+=t_i^j$ if $1+\gamma_i\neq 0$, and $x_i^+=w_i^j$ if $1+\gamma_i=0$. Then $x_i^j=\epsilon_ix_i^++q_i^+$, where $0<\epsilon_i<p$ and $q_i^+\in\langle W_{j-1}U_{j-1}V_{j-1}\setminus\{x_i^+\}\rangle$. $x_1^+,\ldots,x_k^+$ is a basis of $\langle YX_1^+X_2^+X_2^-\ldots X_{j-1}^+X_{j-1}^-X_j^+\rangle$ modulo $\langle YX_1^+X_2^+X_2^-\ldots X_{j-1}^+X_{j-1}^-\rangle$ as $x_1^j,\ldots,x_k^j$. By (S4) again w.l.o.g. $\epsilon_i=1$ and $q_i^+\in\langle W_{j-1}U_{j-1}V_{j-1}\setminus\{x_1^+,\ldots,x_k^+\}\rangle$. First we assume that $Z_j^{+by}\neq\emptyset$. W.l.o.g. $t_1^j\in Z_j^{+by}$. By (S1) we obtain Φ_i^j's that freely generate $\langle YX_1^+X_2^+X_2^-\ldots X_j^+X_j^-\rangle$ over $\langle YX_1^+X_2^+X_2^-\ldots X_{j-1}^+X_{j-1}^-\rangle$. Φ_i^j is of the form

$$\sum_{1\leq s<t\leq k}\alpha_{st}^i\left((x_s^j\wedge x_t^j)-(x_s^1\wedge x_t^1)\right)+\sum_{1\leq t\leq k}(y_t^i\wedge(x_t^j-x_t^1))+\sum_{k<t\leq r}(y_t^i\wedge z_t^j).$$

Then there are some i and s with $\alpha_{1s}^i\neq 0$. If we rewrite Φ_i^j with respect to the suitable basis $W_jU_jV_j$, then $x_1^+\wedge x_s^+$ occurs in the linear combination of basic commutators exactly once (with the coefficient $\alpha_{1s}\neq 0$). If $x_s^+\in A$, then this is clear. Otherwise further $(x_1^+\wedge x_s^+)$'s can be produced by the $(x_l^1\wedge x_q^1)$'s only, since $Y\subseteq A$. But t_1 or w_1 occurs in x_l^1 as t_1+w_1 only and t_s or w_s as t_s+w_s only. Hence this is impossible. As above let Φ_i^j be $\Phi_i^b+\Phi_i^c$, and $J\in\wedge^2A$, be such that $\Phi_i^b+J\in N(B)$ and $\Phi_i^c-J\in N(C)$.

If $x_s^+=t_s^j\in A$, then $x_1^+\wedge x_s^+$ does not occur in $\wedge^2\langle W_{j-1}U_{j-1}V_{j-1}\rangle+\wedge^2A$. If $x_1^+\in U_j$, then $\Delta_{f(j)}=\Phi_i^b+J$ is independent from $\Delta_1,\ldots,\Delta_{j-1}$ modulo $\wedge^2A$. If $x_1^+\in V_j$, then $\Theta_{g(j)}=\Phi_i^c-J$ is independent from $\Theta_1,\ldots,\Theta_{g(j-1)}$ modulo $\wedge^2A$. (11.vi) and (11.vii) follow again by Lemma 4.3.

If x_s^+ is not in A, then by Lemma 4.3 either x_1^+ and x_s^+ are in U_j or they are in V_j. We construct $\Delta_{f(j)}$ or $\Theta_{g(j)}$ respectively as above. The independence of $\Delta_{f(j)}$ from $\Delta_1,\ldots,\Delta_{f(j-1)}$ or of $\Theta_{g(j)}$ from $\Theta_1,\ldots,\Theta_{g(j-1)}$ over $\wedge^2A$ respectively follows, since by (11.vi) and (11.vii) $x_1^+\wedge x_s^+$ cannot occur in the representations of $\Delta_1,\ldots,\Delta_{f(j-1)}$ and of $\Theta_1,\ldots,\Theta_{g(j-1)}$. (11.vi) and (11.vii) follow as above.

There remains the case $Z_j^{+by}=0$. That means $x_s^+=t_s^+\in A$ for all s with $1\leq s\leq k$. We use $x_s^j=x_s^++q_s^+$, where $q_s^+\in\langle W_{j-1}U_{j-1}V_{j-1}\rangle$. Since by assumption $X_j^+\not\subseteq A$, there exists some $w\in U_{j-1}V_{j-1}$ and some t such that for $1\leq s\leq k$ $x_s^j=x_s^++\gamma_sw+q_s^0$. where $0\leq\gamma_s<p$, $q_s^0\in\langle W_{j-1}U_{j-1}V_{j-1}\setminus\{w\}\rangle$, and $\gamma_t\neq 0$. By (S4) w.l.o.g. $t=1$. Note $w\notin Y$, since $Y\subseteq A$. By (S4) w.l.o.g. $\gamma_s=0$ if and only if $s\neq 1$. Choose s such that $\alpha_{1s}\neq 0$ in Φ_i^j. Then $w\wedge x_s^+$ is a new basic commutator in the representation of Φ_i^j with respect to $W_jU_jV_j$. Here we use $w\notin Y$. We obtain $\Delta_{f(j)}$ or $\Theta_{g(j)}$, the linear independence over $\wedge^2A$, and (11.vi,vii) as above. □

Lemma 6.6. *Assume that A, B, and C are in $\mathcal{S}$ and fulfill $(\Sigma 3)$. A is a common substructure of B and C. Let D be $B \star_A C$. Suppose that A is selfsufficient in B and $\dim(B/A)$-selfsufficient in C. Let $Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ be a reduced k-sequence in D.*

> (i) *If there exists some j $(2 \leq j \leq h)$ such that $\langle Y X_1^+ X_j^+ X_j^- \rangle \cap C = \langle Y X_1^+ \rangle \cap C$, then $Y \subseteq B$.*
> (ii) *If there exists some j $(2 \leq j \leq h)$ such that $\langle Y X_1^+ X_j^+ X_j^- \rangle \cap B = \langle Y X_1^+ \rangle \cap B$, then $Y \subseteq C$.*

Proof. Assume that $Y = Y^a Y^b Y^c Y^d$ is a suitable basis of $\langle Y \rangle$, and

$$S = S^a S^{by} S^{bx} S^{cx} S^e$$

is a suitable extension of Y for $\langle Y X_1^+ \rangle$. Then (i) and (ii) have the following form:

> (i) If $Z_j = Z_j^{bx} Z_j^e$ is a suitable extension of YS for $\langle Y X_1^+ X_j^+ X_j^- \rangle$ over $\langle Y X_1^+ \rangle$, then $Y \subseteq B$.
> (ii) If $Z_j = Z_j^{cx} Z_j^e$ is a suitable extension of YS for $\langle Y X_1^+ X_j^+ X_j^- \rangle$ over $\langle Y X_1^+ \rangle$, then $Y \subseteq C$.

By Theorem 4.1, D fulfills $(\Sigma 3)^0$. We show (i); (ii) follows similarly. Assume that Y^d is choosen in such a way that for every y_b in S^{by} there is some y in Y^d with $y = y_b + y_c$, where y_c is in S^{cy}. If $Y \not\subseteq B$, then $Y^c Y^d \neq \emptyset$. Let y be an element of $Y^c Y^d$. We have $y^+ = y_c$ if $y = y_b + y_c$ with $y_b \in S^{by}$ and $y_c \in S^{cy}$; otherwise $y^+ = y$. Using (S2), we can assume that $j = 2$. By (S1), $N(\langle Y X_1^+ X_2^+ X_2^- \rangle)$ is freely generated by $\Phi_1^2, \ldots, \Phi_r^2$ over $N(\langle Y X_1^+ \rangle)$, where the Φ_i^2's are of the form

$$\sum_{1 \leq s < t \leq k} \alpha_{st}^i \left((x_s^2 \wedge x_t^2) - (x_s^1 \wedge x_t^1) \right) + \sum_{1 \leq t \leq k} (y_t^i \wedge (x_t^2 - x_t^1)) + \sum_{k < t \leq r} (y_t^i \wedge z_t^2).$$

If we rewrite the Φ_i^2's with respect to $Y X_1^+ X_2^+ X_2^-$, which means the y_t^i's with respect to the basis Y, then we denote by γ_t^i the coefficient of $y \wedge (x_t^2 - x_t^1)$ for $1 \leq t \leq k$ and of $y \wedge z_t^2$ for $k < t \leq r$.

For $1 \leq t \leq k$ assume $x_t^2 = x_t + x_t^0 + \epsilon_t y^+$, where $x_t \in \langle Z_2^{bx} Z_2^e \rangle$, $x_t^0 \in \langle YS \setminus \{y^+\} \rangle$, and for $k < t \leq r$ assume $z_t^2 = z_t + z_t^0$, where $z_t \in \langle Z_2^{by} Z_2^e \rangle$ and $z_t^0 \in \langle YS \rangle$. W.l.o.g. we suppose that $\{x_1, \ldots, x_k, z_{k+1}, \ldots, z_r\} = Z_2^{by} Z_2^e$ as sets, since otherwise we could apply an appropriate well-tolerated automorphism and use (S4).

In the representation of Φ_i^2 $(1 \leq i \leq r)$ with respect to a suitable basis of D that is consistent with $Y S Z_2^{bx} Z_2^e$ the coefficient of $y^+ \wedge z_t$ is γ_t^i and the coefficient of $y^+ \wedge x_t$ is $\gamma_t^i + \sum_{s < t} \alpha_{st}^i \epsilon_s - \sum_{t < s} \alpha_{ts}^i \epsilon_s$. There has to be some i such that at least one of these coefficients is not zero. Otherwise the well-tolerated automorphism e defined by $e(x_t^j) = x_t^j - \epsilon_t y$ and $e(z_t^j) = z_t^j$ would give a representation of all Φ_i^j as in (S1) with respect to $Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \setminus \{y\}$. But $y^+ \wedge x_t$ or $y^+ \wedge z_t$ with $y^+ \in Y^c Y^d S^{cy}$ and $x_t, z_t \in Z_2^{bx} Z_2^e$ cannot occur in the representation of any Φ_i^j by Lemma 4.3. $\square$

Lemma 6.7. *Assume that we are in the situation of Lemma 6.4.*

Let $Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ be a reduced k-sequence where $0 < k$ and $r, |Y| + k < m$.

> (i) *If $\langle Y X_1^+ \rangle \subseteq B$ and $\langle Y X_1^+ \rangle \cap A = \langle Y \rangle \cap A$, then all $X_j^+ X_j^-$ are in B.*
> (ii) *If $\langle Y X_1^+ \rangle \subseteq C$ and $\langle Y X_1^+ \rangle \cap A = \langle Y \rangle \cap A$, then all $X_j^+ X_j^-$ are in C.*

Proof. We can write the assertions in the following way:

(i) If $Y \subseteq B$ and $\langle Y X_1^+ \rangle$ has a suitable basis $Y Z_1^{+bx}$, then all $X_j^+ X_j^-$ are in B.

(ii) If $Y \subseteq C$ and $\langle Y X_1^+ \rangle$ has a suitable basis $Y Z_1^{+cx}$, then all $X_j^+ X_j^-$ are in C.

We show (i). The proof of (ii) is similar. Let Y be $Y^a Y^b$, a suitable basis of $\langle Y \rangle$, and let $Y Z_1^{+bx} Z_j^a Z_j^{bx} Z_j^{cx} Z_j^e$ be a suitable basis of $\langle Y X_1^+ X_j^+ X_j^- \rangle$. W.l.o.g. $Z_1^{+bx} = X_1^+$. We have to show that $Z_j^{cx} Z_j^e = \emptyset$. Otherwise let us first assume that $Z_j^{cx} Z_j^e \neq \emptyset$ and $Z_j^a Z_j^{bx} \neq \emptyset$. Let Z_j^e be $\{t_1 + w_1, \dots, t_l + w_l\}$. Note that $Y^c Y^d = \emptyset$. By Lemma 5.1 $\delta(\langle \{w_1, \dots, w_l\} Z^{cx} A \rangle) < \delta(A)$, a contradiction to the $2m^2$-selfsufficiency of A in C. Note that $r < m$.

There remains the case $Z_j^a Z_j^{bx} = \emptyset$. Then $Y Z_1^{+bx} Z_j^{+cx} Z_j^{+e}$ is a suitable basis of $\langle Y X_1^+ X_j^+ \rangle$. A suitable basis of $\langle Y X_j^+ \rangle$ has the form $Y S_j^{+cx} S_j^{+e}$. By Lemma 6.3 $S_j^{+e} = Z_j^{+e} = \emptyset$, and so $\langle Y X_1^+ X_j^+ X_j^- \rangle$ has a suitable basis $Y Z_1^{+bx} Z_j^{+cx} Z_j^{-cx} Z_j^{-e}$, where $X_1^+ = Z_1^{+bx}$ and $X_j^+ = Z_j^{+cx}$. By Lemma 6.6 $Y \subseteq C$, and therefore $Y \subseteq A$.

Next we show $Z_j^{-cx} = \emptyset$. We use $\{t_1 + w_1, \dots, t_l + w_l\}$ for Z_j^{-e}, where $t_i \in B$ and $w_i \in C$. Remember that $X_1^+ = Z_1^{+bx}$ and $X_j^+ = Z_j^{+cx}$. By (S3) and (S1) we have that $\langle Y X_j^+ X_1^+ Z_j^{-cx} Z_j^{-e} \rangle$ is a minimal extension of $\langle Y X_j^+ \rangle$. $|Z^{-cy}| < r$, since $0 < k$. If $Z^{-cx} \neq \emptyset$, then by Lemma 5.1 (1) $\dim(N(\langle \{t_1, \dots, t_l\} X_1^+ A \rangle)) - \dim(N(A)) > |X_1^+| + l = k + l$. It follows that $\delta(\langle \{t_1, \dots, t_l\} X_1^+ A \rangle) < \delta(A)$, a contradiction to the selfsufficiency of A in B.

We have shown that

$$Y X_1^+ X_j^+ X_j^-$$

is a suitable basis. We write $Y^a X_1^{+bx} X_j^{+cx} X_j^{-e}$ to indicate the nature of the single parts. As above, by (S1) and (S3) $\langle Y^a X_j^{+cx} X_1^{+bx} X_j^{-e} \rangle$ is a minimal extension of $\langle Y^a X_j^{+cx} \rangle$. By Lemma 5.1 (6) applied to $\langle Y^a X_j^{+cx} X_1^{+bx} X_j^{-e} \rangle$ over $\langle Y^a X_j^{+cx} \rangle$ we obtain

$$\dim(N(\langle X_1^{+bx} \{t_1, \dots, t_{r-k}\} A \rangle)) - \dim(N(A)) \geq r.$$

Hence $\delta(\langle X_1^{+bx} \{t_1, \dots, t_{r-k}\} A \rangle) \leq \delta(A)$. Since B is a minimal extension of A, equality follows, and $\langle X_1^{+bx} \{t_1, \dots, t_{r-k}\} A \rangle = B$.

Let us consider the free generators $\Phi_1^j, \dots, \Phi_r^j$ of $N(\langle Y^a X_1^{+bx} X_j^{+cx} X_j^{-e} \rangle)$ over $N(\langle Y^a X_1^{+bx} \rangle)$ given by (S1). By Lemma 4.3 there are $J_i \in \wedge^2 A$ such that $\Phi_i^j = \Phi_i^c - \Phi_i^b$, $\Phi_i^b + J_i \in N(B)$, and $\Phi_i^c + J_i \in N(C)$, where

$$\Phi_i^b = \sum_{1 \leq t < s \leq k} \alpha_{ts}^i (x_t^1 \wedge x_s^1) + \sum_{1 \leq t \leq k} (y_t^i \wedge x_t^1) + \sum_{k < t \leq r} (y_t^i \wedge -t_{t-k}),$$

$$\Phi_i^c = \sum_{1 \leq t < s \leq k} \alpha_{ts}^i (x_t^j \wedge x_s^j) + \sum_{1 \leq t \leq k} (y_t^i \wedge x_t^j) + \sum_{k < t \leq r} (y_t^i \wedge w_{t-k}).$$

Note that

$$z_{k+l}^j = t_l + w_l.$$

Since $\Phi_1^j, \dots, \Phi_r^j$ are linearly independent over $N(\langle Y^a X_1^{+bx} \rangle)$, every non-trivial linear combination of them contains some basic commutator $x_t^j \wedge x_s^j$ or $a \wedge x_t^j$ or $a \wedge t_s$ for some $a \in Y$. Hence this linear combination also contains $x_t^1 \wedge x_s^1$ or $a \wedge x_t^1$ or $a \wedge w_s$ respectively. Therefore $\Phi_1^c + J_1, \dots, \Phi_r^c + J_r$ are linearly independent over $\wedge^2 A$ and $\Phi_1^b + J_1, \dots, \Phi_r^b + J_r$ are linearly independent over $\wedge^2 A$. Since $B =$

$\langle X_1^{+bx}\{t_1, \ldots, t_{r-k}\}A\rangle$ and $\delta(B) = \delta(A)$, $\Phi_1^b + J_1, \ldots, \Phi_r^b + J_r$ freely generate $N(B)$ over $N(A)$. Since A is $2m^2$-selfsufficient in C, we obtain that $\Phi_1^c, \ldots, \Phi_r^c$ freely generate $N(\langle X_j^{cx}\{w_1, \ldots, w_{r-k}\}A\rangle$ over $N(A)$. Hence $f(x_i^1) = x_i^j$ for $1 \leq i \leq k$, $f(t_i) = -w_i$ for $1 \leq i \leq r - k$, and $f(a) = a$ for $a \in A$ induces an isomorphism of B into C that is the identity on A and can be extended to the corresponding structures in $\mathcal{S}$. This contradicts the assumption of the lemma, that B is not realized in C. □

Proof of Lemma 6.4. Let $YX_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ be a reduced k-sequence in D with $r, |Y| + k < m$. Assume Y is a suitable basis of $\langle Y\rangle$.

Case 1. $k > 0$ and there exists some j such that $\langle YX_j^+\rangle \cap C = \langle Y\rangle \cap C$. By (S3) we can assume w.l.o.g. that $j = 1$.

Case 1.1. There exists an i such that $2 \leq i \leq h$, $\langle YX_1^+ X_i^+ X_i^-\rangle \cap C = \langle YX_1^+\rangle \cap C$.

By Lemma 6.6 $Y \subseteq B$. By (S2) we can assume w.l.o.g. that $i = 2$. By the assumption we have $\langle YX_1^+\rangle \cap C = \langle Y\rangle \cap C$ and $\langle YX_1^+ X_2^+\rangle \cap C = \langle YX_1^+\rangle \cap C$. Lemma 6.3 implies $\langle YX_1^+\rangle = \langle Y\rangle + (\langle YX_1^+\rangle \cap B)$. Using $Y \subseteq B$ we get $\langle YX_1^+\rangle \subseteq B$. Then $\langle YX_1^+\rangle \cap C = \langle Y\rangle \cap C$ implies $\langle YX_1^+\rangle \cap A = \langle Y\rangle \cap A$. Hence by Lemma 6.7 all $X_j^+ X_j^- \subseteq B$. The assertion follows from $(\Sigma 4)^m$ for B.

Case 1.2. There exists an i such that $2 \leq i \leq h$, and $\langle YX_1^+ X_i^+ X_i^-\rangle \cap B = \langle YX_1^+\rangle \cap B$.

By Lemma 6.6 $Y \subseteq C$. By (S2) we can assume w.l.o.g. that $i = 2$. By the assumptions we have $\langle YX_1^+\rangle \cap C = \langle Y\rangle \cap C$, and $\langle YX_1^+ X_2^+\rangle \cap B = \langle YX_1^+\rangle \cap B$. Note that this implies $\langle YX_2^+\rangle \cap B = \langle Y\rangle \cap B$. Lemma 6.3 implies $\langle YX_2^+\rangle = \langle Y\rangle + (\langle YX_2^+\rangle \cap C)$. Using $Y \subseteq C$ we get $\langle YX_2^+\rangle \subseteq C$. Since $\langle YX_2^+\rangle \cap B = \langle Y\rangle \cap B$ we get $\langle YX_2^+\rangle \cap A = \langle Y\rangle \cap A$. By (S3) we can exchange X_1^+ and X_2^+. Hence by Lemma 6.7 we get $X_1^+ \subseteq C$, a contradiction to the general assumption of Case 1.

Case 1.3. For all i with $2 \leq i \leq h$ we have $\langle YX_1^+ X_i^+ X_i^-\rangle \cap C \neq \langle YX_1^+\rangle \cap C$ and $\langle YX_1^+ X_i^+ X_i^-\rangle \cap B \neq \langle YX_1^+\rangle \cap B$.

In this case the assertion follows from Lemma 6.5.

Case 2. The roles of B and C are exchanged in Case 1.

The proof is completely symmetric.

Case 3. $k > 0$ and for every j with $1 \leq j \leq h$ we have neither $\langle YX_j^+\rangle \cap C = \langle Y\rangle \cap C$ nor $\langle YX_j^+\rangle \cap B = \langle Y\rangle \cap B$.

We get $\langle YX_1^+ X_j^+ X_j^-\rangle \cap C \neq \langle YX_1^+\rangle \cap C$ and $\langle YX_1^+ X_j^+ X_j^-\rangle \cap B \neq \langle YX_1^+\rangle \cap B$. The assertion follows from Lemma 6.5.

Case 4. $k = 0$ and for every j with $2 \leq j \leq h$ we have neither $\langle YX_j^-\rangle \cap C = \langle Y\rangle \cap C$ nor $\langle YX_j^-\rangle \cap B = \langle Y\rangle \cap B$.

In this case the assertion follows immediately from Lemma 6.5.

Case 5. $k = 0$ and there is some j with $2 \leq j \leq h$ such that $\langle YX_j^-\rangle \cap C = \langle Y\rangle \cap C$.

In this case $\langle Y X_j^- \rangle$ has a suitable basis $Y S_j^{bx} S_j^e$. By Lemma 6.6 $Y \subseteq B$. By (S2) w.l.o.g. $j = 2$. Let $Y S_i^a S_i^{bx} S_i^{cx} S_i^e$ be a suitable basis of $\langle Y X_i^- \rangle$ for $2 \le i \le h$. Since $Y^d = \emptyset$ there is no S_i^{by}. Let S_i^e be $\{t_1 + w_1, \ldots, t_l + w_l\}$, where $t_1, \ldots, t_l$ are in B, $w_1, \ldots, w_l$ are in C, and $t_1, \ldots, t_l, w_1, \ldots, w_l$ are linearly independent over A. First assume $Y^b \ne \emptyset$. If $S_i^a S_i^{bx} = \emptyset$, then $Y \subseteq C$ by Lemma 6.6, a contradiction to $Y^b \ne \emptyset$. Hence $S_i^a S_i^{bx} \ne \emptyset$. If $S_i^{cx} S_i^e \ne \emptyset$, then $\dim(N(\langle S_i^{cx} \{w_1, \ldots, w_l\} A \rangle)) - \dim(N(A)) > |S_i^{cx}| + l$ by Lemma 5.1. Therefore $\delta(\langle S_i^{cx} \{w_1, \ldots, w_l\} A \rangle) < \delta(A)$, a contradiction to the $2m^2$-selfsufficiency of A in C. Note that $r < m$. We have shown that $S_i^{cx} S_i^e = \emptyset$ for every i. Hence all X_i^- are in B. The assertion follows from $(\Sigma 4)^m$ for B.

There remains the case $Y = Y^a \subseteq A$. A suitable basis $Y S_i^e$ for some $\langle Y X_i^- \rangle$ is impossible. In this case we could assume $X_i^- = S_i^e$ by (S4) and apply Lemma 4.4 to the situation described in (S1). We would get a contradiction to $\delta(\langle Y \rangle) = \delta(\langle Y X_i^- \rangle)$ Then by Lemma 5.2 every X_i^- is in B. The assertion follows from $(\Sigma 4)^m$ for B.

Case 6. $k = 0$ and there is some j with $2 \le j \le h$ such that $\langle Y X_j^- \rangle \cap B = \langle Y \rangle \cap B$.

A similar proof as for Case 5 works. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 6.8. *Assume A, B, and C are finite structures in $\mathcal{S}$ that satisfy $(\Sigma 3)$ and $(\Sigma 4)$. Furthermore A is a common substructure of B and C, B is a minimal extension of A that is not realized in C, and A is n-selfsufficient in C, where $n \ge 2(2 \dim(B) + 1)^2$. Let D be $B \star_A C$.*

Then C is selfsufficient in D, B is n-selfsufficient in D, and D satisfies $(\Sigma 3)$ and $(\Sigma 4)$.

Proof. By Theorem 4.1 we have everything except $(\Sigma 4)$ for D. Note that

$$2(2 \dim(B) + 1)^2 \ge \dim(B/A) + 2.$$

By Lemma 6.4 we obtain $(\Sigma 4)^{2 dim(B) + 1}$. That means for every reduced k-sequence with $|Y| + k, r \le 2 \dim(B)$ we have $h \le 2 max\{r, |Y| + k\} + 1$.

Now we consider reduced k-sequences $Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ with $r > 2 \dim(B)$ or $|Y| + k > 2 \dim(B)$. Assume that Y is a suitable basis of $\langle Y \rangle$. We choose

$$Z_j = Z_j^a Z_j^{by} Z_j^{bx} Z_j^{cx} Z_j^e$$

such that $Y Z_1 \ldots Z_h$ gives us a suitable basis for $\langle Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^- \rangle$ that respects $\langle Y \rangle$, $\langle Y X_1^+ \rangle, \ldots, \langle Y X_1^+ X_2^+ X_2^- \ldots X_{h-1}^+ X_{h-1}^- \rangle$.

If $h \le 2 \dim(B) + 1$, then $(\Sigma 4)$ is fulfilled by this sequence. In the case $h > 2 \dim(B) + 1$ we will show that the whole reduced k-sequence is contained in C and the assertion follows from $(\Sigma 4)$ for C.

If $h > 2 \dim(B) + 1$, then there are different j_0, j_1 with $1 \le j_0 < j_1 \le 2 \dim(B) + 2$, such that

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j_0}^+ X_{j_0}^- \rangle \cap B = \langle Y X_1^+ X_2^+ X_2^- \ldots X_{j_0-1}^+ X_{j_0-1}^- \rangle \cap B$$

and

$$\langle Y X_1^+ X_2^+ X_2^- \ldots X_{j_1}^+ X_{j_1}^- \rangle \cap B = \langle Y X_1^+ X_2^+ X_2^- \ldots X_{j_1-1}^+ X_{j_1-1}^- \rangle \cap B.$$

After an appropriate application of (S2) and (S3) we can assume that $j_0 = 1$ and $j_1 = 2$. Then $\langle Y X_1^+ \rangle \cap B = \langle Y \rangle \cap B$ and $\langle Y X_1^+ X_2^+ X_2^- \rangle \cap B = \langle Y X_1^+ \rangle \cap B$. Hence by Lemma 6.6 $Y \subseteq C$. Furthermore, $\langle Y X_1^+ X_2^+ \rangle \cap B = \langle Y \rangle \cap B$. By Lemma 6.3 $\langle Y X_1^+ X_2^+ \rangle = \langle Y \rangle + (\langle Y X_1^+ X_2^+ \rangle \cap C)$. The last two equations imply that $\langle Y X_1^+ X_2^+ \rangle$

has a suitable basis $YZ_1^{+cx}Z_2^{+cx} \subseteq C$, where YZ_1^{+cx} is a suitable basis of $\langle YX_1^+ \rangle$. Now we know that $Y \subseteq C$ and $Z_1 = Z_1^{+cx}$. Choose j minimal such that $Z_j \not\subseteq C$. Then $YS_1^{+cx}Z_2 \ldots Z_j$ is a suitable basis of $\langle YX_1^+X_2^+X_2^- \ldots X_j^+X_j^- \rangle$. Let Z_j be $Z_j^a Z_j^{bx} Z_j^{cx} Z_j^e$. Note that $Z_j^{by} = \emptyset$ and $Z_j^{bx} Z_j^e \neq \emptyset$, by the choice of j.

First assume $Z_j^a Z_j^{cx} \neq \emptyset$. Note that $\dim(B) < r$ implies $Z_j^{cx} \neq \emptyset$. By Lemma 5.1 (1) we obtain $\dim(N(\langle Z_j^{bx}\{t_1, \ldots t_l\}A\rangle)) - \dim(N(A)) > |Z_j^{bx}| + l$, where $Z_j^e = \{t_1+w_1, \ldots, t_l+w_l\}$ and $Z_j^{bx}\{t_1, \ldots, t_l\} \subseteq B$ is linearly independent over A. Then $\delta(\langle Z_j^{bx}\{t_1, \ldots, t_l\}A\rangle) < \delta(A)$, a contradiction to the selfsufficiency of A in B.

If $Z_j^a Z_j^{cx} = \emptyset$, then $r \leq \dim(B)$. Hence by the assumption $|Y| + k > 2\dim(B)$. Since $k \leq r \leq \dim(B)$, we get $|Y| > \dim(B)$ and therefore $Y^c \neq \emptyset$. On the other hand, by Lemma 6.6 we have $Y \subseteq B$, a contradiction to $Y^c \neq \emptyset$. We have shown that all $X_j^+X_j^-$ are in C. The assertion follows from ($\Sigma 4$) for C. $\qquad \square$

The final theorem summarizes the efforts of this section and formulates the result for further applications.

Theorem 6.9 (Second Amalgamation Theorem). *Assume that A, B, and C are finite structures in $\mathcal{S}$ that satisfy ($\Sigma 3$) and ($\Sigma 4$). Suppose that A is a common substructure of B and C, A is selfsufficient in B, and A is $(\dim(B/A)+n)$-selfsufficient in C, where $n \geq 2(2\dim(B) + 1)^2$.*

Then there exists an amalgam D of B and C over A such that D fulfills ($\Sigma 3$) and ($\Sigma 4$), C is selfsufficient in D, and B is n-selfsufficient in D. If A is selfsufficient in C, then B is selfsufficient in D.

Proof. We decompose B into a nonrefinable tower $A = B_0 \subset B_1 \subset \ldots \subset B_{n-1} \subset B_n = B$ of subspaces such that each B_i is selfsufficient in B.

First we assume that we have shown the assertion for all cases where $n = 1$. We prove the theorem by induction on n. By induction hypothesis there exists an amalgam D_{n-1} of B_{n-1} and C over A such that D_{n-1} satisfies ($\Sigma 3$) and ($\Sigma 4$), C is selfsufficient in D_{i-1}, and B_{n-1} is $(\dim(B_n/B_{n-1})+n)$-selfsufficient in D_{n-1}. Now we apply the assertion for the case $n = 1$ to the following situation: B_{n-1} plays the role of A, B plays the role of B, and D_{n-1} plays the role of C. The D we obtain has the desired properties. C is selfsufficient in D by Lemma 3.10.

Now we consider the case $n = 1$. Then $\delta(A) \leq \delta(B) \leq \delta(A) + 1$. If $\delta(A) = \delta(B)$, then B is a minimal extension of A. If there is a realization B^+ of B in C over A, then take $D = C$. The assertion follows. Note that B^+ is n-selfsufficient in C, since $\delta(B) = \delta(A)$ and A is $(\dim(B/A) + n)$-selfsufficient in C. If there is no realization of B in C, then we apply Lemma 6.8.

There remains the case $\delta(B) = \delta(A) + 1$. Then $\dim(B) = \dim(A) + 1$. Again we take D as $B \star_A C$. By Theorem 4.1 we have ($\Sigma 3$) for D, selfsufficiency of C in D, and n-selfsufficiency of B in D. We show ($\Sigma 4$). Note that for every $b \in B \setminus A$ we obtain $\langle A\{b\}\rangle = B$. Furthermore, $D = \langle C\{b\}\rangle$ for every $b \in D \setminus C$. Let $YX_1^+X_2^+X_2^- \ldots X_h^+X_h^-$ be a reduced k-sequence in D. Our aim is to prove that the sequence lies completely in C. Then ($\Sigma 4$) for C implies ($\Sigma 4$) for D. If $Y \not\subseteq C$, then in (S1) there is some

$$\Phi_i^2 \in N(\langle YX_1^+X_2^+X_2^- \rangle) \setminus N(\langle YX_1^+ \rangle)$$

such that $y_t^i \notin C$ for some t. By (S4) we can consider a suitable basis $\{y_t^i\}Y^0Z_1Z_2$ in D, where $\{y_t^i\}Y^0 = Y$ and $Y^0, Z_1, Z_2 \subseteq C$, $X_1^+ \subseteq \langle \{y_t^i\}Z_1 \rangle$, and $X_2^+X_2^- = Z_2$. If we represent Φ_i^2 with respect to this basis, we obtain a non-vanishing summand

$(y_t^i \wedge x_t^2)$ or $(y_t^i \wedge z_t^2)$. This is the only occurrence of this basic commutator. We obtain a contradiction, since in D no element of $D \setminus C$ is involved in a relation. The same argument works for $Y \subseteq C$ and $x_i^j \notin C$ or $z_i^j \notin C$ for some i and j. If $x_i^j \notin C$, then by (S3) and (S4) w.l.o.g. $j = i = 1$. Let $Y\{x_1^1\}Z_1 Z_2$ be a basis of $\langle Y X_1^+ X_2^+ X_2^- \rangle$, where $Z_1, Z_2 \subseteq C$, and $Y\{x_1^1\}Z_1$ is a basis of $\langle Y X_1^1 \rangle$. By (S4) w.l.o.g. $\{x_1^1\}Z_1 = X_1^+$. In (S1) there exists some Φ_i^2 with $\alpha_{1s}^i \neq 0$ for some s. If we rewrite Φ_i^2 with respect to the basis above, then it contains $x_1^1 \wedge x_s^1$ exactly once with the coefficient α_{1s}^i. Again we have a contradiction. If $Y \subseteq C$ and all $X_j^+ \subseteq C$, but $z_i^j \notin C$, then by (S2) and (S4) w.l.o.g. $j = 2$, $i = k + 1$, and $z_{k+l}^2 \in C$ for $2 \leq l \leq r - k$. Choose Φ_i^2 with $y_{k+1}^i \neq 0$. Then the representation of Φ_i^2 with respect to $Y X_1^+ X_2^+ X_2^-$ contains $y_{k+1}^i \wedge z_{k+1}^2$ exactly once. Since we can assume that $y_{k+1}^i \in Y$, we obtain the desired contradiction. $\qquad\square$

7. Construction of the Group

We now come back to our category of groups, that is to the category $\mathcal{G}$. Let H be a subgroup of G, with $G, H \in \mathcal{G}$. Then we say that H is selfsufficient in G if this is true for the corresponding structures in $\mathcal{S}$.

Now let Σ be the following elementary theory formulated in the pure group language. Let G be any model of Σ and let M be the corresponding structure in $\mathcal{S}$. (We still have to show its existence.) The axioms of Σ express the following:

(Σ1): G is a nilpotent group of class 2 and of exponent $p > 2$. p is a fixed prime.

(Σ2): $\forall x \notin Z(G) \forall z \in Z(G) \exists y ([x, y] = z)$.

(Σ3): For every finite subspace H of the corresponding $\mathcal{S}$-structure M we have $\delta(H) = dim(H)$ if $dim(H) \leq 3$ and $\delta(H) \geq 3$ otherwise.

(Σ4): For every reduced k-sequence $Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ in the corresponding structure M of $\mathcal{S}$ we have $h \leq 2 max(k + |Y|, |X_2^+| + |X_2^-|) + 1$.

(Σ5): Let A and B be finite $\mathcal{S}$-structures that satisfy (Σ3) and (Σ4). Suppose that A is selfsufficient in B and $(dim(\overline{B}/\overline{A}) + n)$-selfsufficient in M, where $n \geq 2(2 dim(\overline{B}) + 1)^2$. Then there exists an embedding of B in M that extends the embedding of A in M, such that the image of B is n-selfsufficient in G.

Note that axioms (Σ3)-(Σ5) are to be expressed in the language of groups, but refer to the properties of the corresponding structure M in the category $\mathcal{S}$. Using the Second Amalgamation Theorem, we will prove the consistency, the completeness, and the uncountable categoricity of Σ.

Lemma 7.1. *Assume that M is in $\mathcal{S}$ with the properties (Σ3) and (Σ4). Let g be in M and $b \in \wedge^2 M$ be such that there is no h in M with $(g \wedge h) - b \in N(M)$. Define M^+ in the following way: $M^+ = \langle a \rangle \oplus M$ and $N(M^+) = \langle (g \wedge a) - b \rangle \oplus N(M)$*

   Then M is selfsufficient in M^+, and M^+ satisfies (Σ3) and (Σ4).

Proof. We obtain $\delta(M) = \delta(M^+)$ by construction. Therefore M is selfsufficient in M^+. From the assumption that $(g \wedge x) - b \in N(M)$ has no solution it follows that $\delta(M) \geq 3$.

Next we show (Σ3). Let $K \subseteq M^+$. If $K \subseteq M$, then we are done. Therefore suppose that $K \not\subseteq M$. Let K_1 be $K \cap M$. Note that M^+ can be defined using any $a + h$, where $h \in M$, and the relation $(g \wedge (a + h)) - (b + (g \wedge h))$ instead of $(g \wedge a) - b$. Also $(g \wedge x) - (b + (g \wedge h)) \in N(M)$ has no solution in M. Now $\delta(K) = \delta(K_1) + 1 - dim(N(K)/N(K_1))$. By definition $dim(N(K)/N(K_1)) \leq 1$. If

$dim(K_1) \geq 3$, then the assertion follows. If $dim(K_1) \leq 2$, then $dim(N(K_1)) = 0$ and the only possibility to obtain a relation in $N(K)$ is the following: As noted above, w.l.o.g. $K = \langle a, g_1, g_2 \rangle$, where $g_1, g_2 \in M$ and are linearly independent. Then w.l.o.g. $g_1 = g$. A relation would have the form $(g \wedge a) - (g \wedge rg_2) \in N(K)$. This would imply a solution of $(g \wedge x) - b \in N(M)$, a contradiction. ($\Sigma 3$) is shown.

To show ($\Sigma 4$) let $Y X_1^+ X_2^+ X_2^- \ldots X_h^+ X_h^-$ be a reduced k-sequence that is not completely in M. Note that $r = 1$ is impossible, since by ($\Sigma 3$) relations of the form $y \wedge z$ do not exist.

First we assume that $Y \nsubseteq M$. Then w.l.o.g. $a \in Y$, and $Y \setminus \{a\} \subseteq M$. We fix some j with $2 \leq j \leq h$ and consider the Φ_i^j's in (S1):

$$\sum_{1 \leq s < t \leq k} \alpha_{st}^i \left((x_s^j \wedge x_t^j) - (x_s^1 \wedge x_t^1) \right) + \sum_{1 \leq t \leq k} (y_t^i \wedge (x_t^j - x_t^1)) + \sum_{k < t \leq r} (y_t^i \wedge z_t^j).$$

After the application of an appropriate well-tolerated automorphism we can assume that $X_j^+ \subseteq M$. Since all of Y must be used in (S1), there is some i_j such that $y_t^{i_j} \neq 0$ for some t with the property that $y_t^{i_j}$ involves a in its representation as a linear combination over Y. Now we rewrite $\Phi_{i_j}^j$ with respect to a basis $Y Z_1^+ X_j^+ Z_j^-$, where $Z_1^+ X_j^+ Z_j^- \subseteq M$, Z_1^+ is a basis of $\langle X_1^+ \rangle$ modulo $\langle \{a\} \rangle$, and Z_j^- is a basis of $\langle X_j^- \rangle$ modulo $\langle \{a\} \rangle$. Then $\Phi_{i_j}^j = (a \wedge u_j) + (a \wedge v_j) + w_j$, where $u_j \neq 0$, $u_j \in \langle X_j^+ Z_j^- \rangle$, $v_j \in \langle Y \setminus \{a\} Z_1^+ \rangle$, and $w_j \in \wedge^2 \langle (Y \setminus \{a\}) Z_1^+ X_j^+ Z_j^- \rangle$. Then $u_j \neq 0$, $u_j + v_j \neq 0$, and $\Phi_{i_j}^j$ has to be a free generator of $N(M^+)$ over $N(M)$. That means $\Phi_{i_j}^j$ is of the form $r(a \wedge g) - b_1$, where $0 < r < p$ and $b_1 \in \wedge^2 M$. It follows that $h \leq 2$, since it is not possible that $u_2 + v_2, u_3 + v_3 \in \langle g \rangle$.

Now we can assume that $Y \subseteq M$. Next we consider the case $X_j^+ \subseteq M$ for some j. By (S3) w.l.o.g. $X_1^+ \subseteq M$. The case $k = 0$ is included. By assumption and by (S2) w.l.o.g. $X_2 \nsubseteq M$. We replace X_2 by a basis aW such that $\langle X_2 \rangle = \langle aW \rangle$ and $W \subseteq M$. Since $\langle Y X_1^+ X_2 \rangle$ is a minimal extension of $\langle Y X_1^+ \rangle$ and $|X_2| = r > 1$, there are $\Phi_{i_1}^2, \Phi_{i_2}^2 \in N(\langle Y X_1^+ X_2 \rangle)$ linearly independent over $N(\langle Y X_1^+ W \rangle)$. This contradicts the assumption that $N(M^+)$ is generated by a single relation over $N(M)$.

Finally, we assume $Y \subseteq M$ but all $X_j^+ \nsubseteq M$. We consider $j > 1$. Using (S4), we can suppose w.l.o.g. that $x_1^j = a$ and $x_2^j, \ldots, x_k^j \in M$. Then there is some i_j such that $\alpha_{1s}^{i_j} \neq 0$ for some s in $\Phi_{i_j}^j$. Hence $\Phi_{i_j}^j$ contains a basic commutator $a \wedge x_s^j$ if we rewrite $\Phi_{i_j}^j$ with respect to a basis $Y Z_1^+ X_j^+ Z_j^-$ where $Z_1^+ = X_1^+$ modulo $\langle a \rangle$ and $Z_j^- = X_j^-$ modulo $\langle a \rangle$. Hence $\Phi_{i_j}^j \notin N(Y X_1^+ X_2^+ X_2^- \ldots X_{j-1}^+ X_{j-1}^-) + N(M)$. Then $\Phi_{i_2}^2$ and $\Phi_{i_3}^3$ are linearly independent over $N(M)$. Hence $h \leq 2$, as desired. $\square$

Let $F_2(p, \kappa)$ be the free group in the variety of all nilpotent groups of class two and with exponent p that has κ-many free generators. Let $F(3)$ be the $\mathcal{S}$-structure that corresponds to $F_2(p, 3)$.

Corollary 7.2. *There are infinitely many finite extensions B_i of $A = F(3)$ such that B_i satisfies ($\Sigma 3$) and ($\Sigma 4$), A is selfsufficient in B_i, and $\delta(A) = \delta(B_i) = 3$.*

Proof. Let a_1, a_2, a_3 be three free generators of A. Define B_1 as M^+ in Lemma 7.1 using the relation $[a_3, x] = [a_1, a_2]$. Let a_4 be the solution in B_1. To get B_2 we use B_1 and $[a_4, x] = [a_2, a_3]$. Let a_5 be the solution in B_2. In general B_i is constructed from B_{i-1} and $[a_{i+2}, x] = [a_i, a_{i+1}]$ with a solution a_{i+3}. The assertion follows from Lemma 7.1. $\square$

Note. Let G be any group that satisfies $(\Sigma 1)$, $(\Sigma 2)^w$, and $(\Sigma 3)$. If A is a subspace of $\overline{G}$ of linear dimension 3, then A is selfsufficient in $\overline{G}$ and A generates a subgroup isomorphic to $F_2(p, 3)$.

Lemma 7.3. Σ *is consistent.*

Proof. We construct an $\mathcal{S}$-structure M that fulfills $(\Sigma 3)$, $(\Sigma 4)$, and $(\Sigma 5)$. Let G be the corresponding group in $\mathcal{G}$. First we show that G fulfills $(\Sigma 2)$. We assume that $(\Sigma 2)$ is not true in G. Then there are g in M and b in $\wedge^2 M$ such that $g \wedge x = b$ has no solution in M. Let A be a finite selfsufficient substructure of M that contains g and b. Of course A fulfills $(\Sigma 3)$ and $(\Sigma 4)$. By Lemma 7.1 there is a finite minimal extension B of A such that B fulfills $(\Sigma 3)$ and $(\Sigma 4)$, and there exists some a in B with $g \wedge a = b$. By axiom $(\Sigma 5)$ we can find B and therefore a in M, a contradiction to the assumption.

Let $(\Sigma 5)'$ be the following consequence of $(\Sigma 5)$ in an ω-saturated model G of Σ. Let M be the corresponding $\mathcal{S}$-structure:

$(\Sigma 5)'$**:** Let A and B be finite $\mathcal{S}$-structures that fulfill $(\Sigma 3)$ and $(\Sigma 4)$. Let A be selfsufficient in B. If A is selfsufficient in M, then there exists a selfsufficient embedding of B in M that extends the embedding of A in M.

Note that $(\Sigma 5)'$ is not elementary. Next we show that every M that satisfies $(\Sigma 3)$, $(\Sigma 4)$, and $(\Sigma 5)'$ fulfills $(\Sigma 5)$ also. To prove $(\Sigma 5)$ let A be selfsufficient in B. Furthermore let A be a $(dim(B/A) + n)$-selfsufficient substructure of M, where $n \geq 2(2dim(B) + 1)^2$. Let C be the finite subspace $CSS(A)$ of M. By Theorem 6.9 there is an amalgam D of B and C over A such that C is selfsufficient in D and B is n-selfsufficient in D. If D plays the role of B and C the role of A in $(\Sigma 5)'$, then we have a selfsufficient embedding of D in M, that extends the embedding of C in M. This gives the desired embedding of B in M with n-selfsufficiency.

To get the desired model G of Σ we show that there is an $\mathcal{S}$-structure M with the properties $(\Sigma 3)$, $(\Sigma 4)$ and $(\Sigma 5)'$. We construct M by successive amalgamation (Theorem 6.9). This is the usual procedure. Let $\{A_i \subset B_i : i < \omega\}$ be an enumeration of all pairs of finite structures in $\mathcal{S}$ with $(\Sigma 3)$ and $(\Sigma 4)$ such that A_i is selfsufficient in B_i, and $dim(B_i) \leq dim(B_j)$ for $i < j$. By Corollary 7.2 there are infinitely many such pairs. Note that in any M with $(\Sigma 3)$ every substructure generated by three elements is selfsufficient . Let C_0 be $F(3)$. If C_{j-1} is constructed with $(\Sigma 3)$ and $(\Sigma 4)$, then consider all pairs $A_i \subset B_i$ with $dim(B_i) \leq j + 2$. Define $D_0 = C_{j-1}$. If D_{i-1} is constructed with the properties $(\Sigma 3)$, $(\Sigma 4)$, and C_{j-1} is selfsufficient in D_{i-1}, then D_i is defined in the following way. Let $A_i^1, \ldots, A_i^s$ be a list of all copies of A_i that are selfsufficient in C_{j-1}. They are also selfsufficient in D_{i-1}. We apply Theorem 6.9 and obtain the amalgam D_i^1 of D_{i-1} and B_i over A_i^1 that again fulfills $(\Sigma 3)$ and $(\Sigma 4)$. D_{i-1} is selfsufficient in D_i^1. The selfsufficient image A_i^1 of A_i in D_{i-1} can be extended to a selfsufficient embedding of B_i into D_i^1. In the next step we apply Theorem 6.9 to D_i^1 and B_i over A_i^2. We get D_i^2 with $(\Sigma 3)$ and $(\Sigma 4)$ such that D_i^1 is selfsufficient in D_i^2 and there is a selfsufficient embedding of B_i in D_i^2 over A_i^2. We iterate the argument and obtain D_i as D_i^s. Then we start to work with $A_{i+1} \subset B_{i+1}$, if $dim(B_{i+1}) \leq j + 2$. If we continue in this way for all pairs $A_l \subset B_l$ with $dim(B_l) \leq j + 2$, then we obtain C_j. It is finite and has the properties $(\Sigma 3)$ and $(\Sigma 4)$. C_{j-1} is selfsufficient in C_j. For every pair $A_i \subset B_i$ with $dim(B_i) \leq j + 2$ and for every selfsufficient copy of A_i in C_{j-1} there is a selfsufficient embedding of B_i in C_j that continues the embedding of A_i. Then

$\bigcup_{j<\omega} C_j$ is the desired $\mathcal{S}$-structure such that the corresponding group is a model of Σ. $\square$

Lemma 7.4. Σ *is complete.*

Proof. Let G_1 and G_2 be two ω-saturated models of Σ. Let M_i be the $\mathcal{S}$-structure corresponding to G_i ($i = 1, 2$). We use a_i and b_i to denote sequences in G_1 and G_2 respectively. We assume that the elements of these sequences are linearly independent over the center. Let $\overline{a_i}$ and $\overline{b_i}$ be the corresponding sequences in M_1 and M_2 respectively. We show that player II has a winning strategy in the infinitely long Ehrenfeucht-Fraïssé game. Let a_0 be a sequence that contains one element c_0 of $G_1 \setminus Z(G_1)$, and b_0 a sequence that contains one element d_0 in $G_2 \setminus Z(G_2)$. $\langle a_0 \rangle$ and $\langle b_0 \rangle$ are selfsufficient in G_1 and G_2 respectively, and there is an isomorphism f_0 of these two subgroups with $f_0(a_0) = b_0$. After the $(i-1)^{th}$ round there are sequences a_{i-1} in G_1 and b_{i-1} in G_2 of linearly independent elements modulo Z and a map f_{i-1} corresponding to the choice of the players that can be extended to an isomorphism of the subgroups generated by a_{i-1} and b_{i-1} respectively. We require furthermore that $\langle a_{i-1} \rangle$ and $\langle b_{i-1} \rangle$ are selfsufficient in G_1 and G_2 respectively. Note that it is very important that the players can choose all elements linearly independent modulo Z. Now we describe the strategy of player II in the ith round. Assume w.l.o.g. that player I has chosen c_i in G_1. W.l.o.g. c_i is not an element of the subgroup generated by a_{i-1}. If c_i is not linearly independent from a_{i-1} modulo Z, then we can assume that $c_i \in Z(G_1)$. By axiom $(\Sigma 2)$ there is some e_i such that $[c_0, e_i] = c_i$. Then e_i is linearly independent from a_{i-1} modulo Z, since otherwise c_i is in the subgroup generated by a_{i-1}. In this case player II can assume that e_i was chosen. So w.l.o.g. we can suppose that c_i is linearly independent from a_{i-1} modulo Z. Let a_i be a sequence of elements linearly independent modulo Z extending $a_{i-1}c_i$ such that its image in $\overline{G}$ is a basis of $CSS(\langle \overline{a}_{i-1}\overline{c}_i \rangle)$. Player II considers $\langle a_i \rangle \setminus \langle a_{i-1} \rangle$ as the sequence of new elements chosen by player I. He applies axiom $(\Sigma 5)$ to M_2, where $\langle \overline{a_{i-1}} \rangle$ plays the role of A and $\langle \overline{a_i} \rangle$ plays the role of B. For every sufficiently large n he gets an n-selfsufficient embedding $\overline{f_i}$ of $\langle \overline{a_i} \rangle$ into M_2, where $\overline{f_i}$ extends the isomorphism of $\langle \overline{a_{i-1}} \rangle$ onto $\langle \overline{b_{i-1}} \rangle$ induced by f_{i-1}. By Lemma 3.2 we can lift $\overline{f_i}$ to an isomorphism f_i of $\langle a_i \rangle$ into G_2 that extends f_{i-1}. By the ω-saturation of G_2 we can find f_i in such a way that $\langle f_i(a_i) \rangle$ is selfsufficient in G_2. We have the desired properties if we denote $f(a_i)$ by b_i in an appropriate way. $\square$

If $\langle a_0 \rangle$ and $\langle b_0 \rangle$ in the proof above are any isomorphic selfsufficient subgroups with $(\Sigma 2)^w$, then the same proof works. If A and B are isomorphic selfsufficient subgroups of the same model G of Σ, then we can extend G to an ω-saturated elementary extension $G_1 = G_2 = G^+$. Then A and B are selfsufficient in G^+ too. We have:

Lemma 7.5. *If A and B are isomorphic finite selfsufficient subgroups of a model of Σ that satisfy $(\Sigma 2)^w$, then they have the same elementary type over the empty set.* $\square$

To prove the uncountable categoricity of Σ we show that for every model G the geometrical closure cl and the algebraic closure acl coincide in $\overline{G}$. It was to ensure this that the axiom $(\Sigma 4)$ was introduced and the Second Amalgamation Theorem was proved.

Lemma 7.6. *Let G be a model of Σ. Let A be a finite subgroup of G with the property $(\Sigma 2)^w$. Let $\overline{A}$ be its image in $\overline{G}$.*

(i) *For all a in $\overline{G}$ we have $a \in acl(\overline{A})$ if and only if $a \in cl(\overline{A})$.*
(ii) *For all elements a and b in G whose images in $\overline{G}$ are not in $acl(A)$, we obtain $t(a/A) = t(b/A)$.*

Proof. Since $CSS(\overline{A}) \subseteq acl(\overline{A})$ we can assume that A is selfsufficient in G.

ad(i). ($\leftarrow$). There is a non-refinable tower $\overline{A} = \overline{B}_0 \subset \overline{B}_1 \subset \overline{B}_2 \ldots \subset \overline{B}_n = CSS(\langle \overline{A}\{a\}\rangle)$ such that $\overline{B}_{i+1}$ is a minimal extension of $\overline{B}_i$ for all $0 \le i \le n-1$. It is sufficient to show that $\overline{B}_{i+1} \subseteq acl(\overline{B}_i)$ for all these i. Otherwise there are infinitely many realizations of the minimal extension $\overline{B}_{i+1}$ over $\overline{B}_i$. Note that all $\overline{B}_i$ are selfsufficient in $\overline{G}$. Therefore by Lemma 5.3 there are infinitely many realizations of the minimal extension $\overline{B}_{i+1}$ over $\overline{B}_i$, that are in free composition over $\overline{B}_i$. This is impossible by $(\Sigma 4)$, as mentioned in Lemma 6.2 4).

($\rightarrow$). If $a \notin cl(\overline{A})$, then $d(\langle \overline{A}\{a\}\rangle) > d(\overline{A})$ and by Lemma 3.8 $d(\langle \overline{A}\{a\}\rangle) = d(\overline{A}) + 1$. Since $\overline{A}$ is selfsufficient in $\overline{G}$, we have that $\langle \overline{A}\{a\}\rangle$ is selfsufficient in $\overline{G}$. If $d(\overline{A}) \le 2$, then every $c \notin \overline{A}$ is isomorphic to a over $\overline{A}$ with respect to the full group structure. By Lemma 7.5, $t(a/A) = t(c/A)$. Therefore $a \notin acl(\overline{A})$. If $d(\overline{A}) \ge 3$, then by Corollary 7.2 $cl(\overline{A})$ is infinite. For each $c \in cl(\overline{A})$ we have $d(\langle \overline{A}\{a+c\}\rangle) = d(\overline{A}) + 1$. Then $a + c$ is isomorphic to a over $\overline{A}$ with respect to the full group structure, and $\langle \overline{A}\{a+c\}\rangle$ and $\langle \overline{A}\{a\}\rangle$ are selfsufficient in $\overline{G}$. By Lemma 7.5 again $t(a+c/A) = t(a/A)$ for infinitely many c. We obtain $a \notin acl(\overline{A})$, as desired.

ad(ii). As above we have A selfsufficient in G. By (i), $d(\langle \overline{A}\{a\}\rangle) = d(\overline{A}) + 1$ and $d(\langle \overline{A}\{b\}\rangle) = d(\overline{A}) + 1$. It follows that a and b are isomorphic over A. By Lemma 7.5 we obtain the assertion. $\qquad\square$

Theorem 7.7. *Σ is uncountably categorical. If G is a model of Σ, then $G/Z(G)$ and $Z(G)$ are strongly minimal. Σ has Morley rank 2. Σ is not countably categorical. Σ has a non-locally-modular geometry.*

Proof. We consider Σ. By Lemmas 7.3 and 7.4, Σ is consistent and complete. By Corollary 7.2 and Lemma 7.6, $acl(\{a, b, c\})$ is infinite for elements a, b, c that are linearly independent over $Z(G)$. It follows that Σ is not countably categorical. To show the uncountable categoricity let G be any model of Σ embedded in the monster model $\mathbf{C}$. Let $a \in \mathbf{C} \setminus G$. If $ah^{-1} \in Z(\mathbf{C})$ for some $h \in G$ and g is any element of G not in $Z(G)$, then by $(\Sigma 2)$ there exists $c \in \mathbf{C} \setminus G + Z(\mathbf{C})$ such that $ah^{-1} = [g, c]$. Therefore every $t(a/G)$ with a formula $Z(xh^{-1})$ is algebraic over a type $t(c/G)$ with $\neg Z(xh^{-1})$ for all $h \in G$. By Lemma 7.6 there exists only one non-algebraic type $t(a/G)$ with $\neg Z(xh^{-1})$ for all $h \in G$. The uncountable categoricity of T follows. Furthermore, $G/Z(G)$ is strongly minimal by Lemma 7.6. By $(\Sigma 2)$ and $(\Sigma 3)$, for any $g \in G \setminus Z(G)$ the map $[g, x]$ is one-to-one from $G/\langle Z(G) \cup \{g\}\rangle$ onto $Z(G)$. Hence $Z(G)$ is strongly minimal. Since for uncountably categorical theories Morley rank and Lascar rank coincide, we obtain Morley rank 2 for Σ by the Berline-Lascar inequalities for the Lascar rank of groups. By [13] Σ has a non-locally-modular geometry, since the models of Σ are not Abelian by finite. It is easy to prove this directly. $\qquad\square$

8. CM-TRIVIALITY

Let Σ be the theory constructed above. In this section we prove the non-interpretability of a field in Σ. The first proof of this result was based on the investigation of the group-configuration of a group interpreted in Σ. It was shown that such a group has to be nilpotent by finite and of bounded exponent. (See [6], section 7.) Here we prove CM-triviality of Σ, following a suggestion of E. Hrushovski and A. Pillay. This geometrical property of a stable theory implies the non-interpretability of a field. Hence Σ does not allow the interpretation of a field, as desired. Furthermore, Pillay [21] has shown that it is impossible to interpret a simple group in an ω-stable theory of finite Morley rank with CM-triviality. Hence no simple group is interpretable in Σ. By Theorem 2.1 we also obtain that every group interpretable in Σ is nilpotent by finite, and of bounded exponent up to an Abelian direct summand.

CM-triviality was introduced by E. Hrushovski in [12]. A stable theory T is CM-trivial if the following three equivalent conditions concerning T^{eq} are fulfilled.

$(CMT1)$**:** Suppose that B_1, B_2, and E are algebraically closed, B_1 and B_2 are independent over E, $acl(B_1B_2) \cap acl(EB_i) = B_i$, and $B_i \cap E = A$. Then B_1 and B_2 are independent over A.

$(CMT2)$**:** If E is algebraically closed and C_1 and C_2 are independent over E, then C_1 and C_2 are independent over $acl(C_1C_2) \cap E$.

$(CMT3)$**:** Let A and B be algebraically closed. Assume $acl(Ac) \cap B = A$. Then $Cb(c/A) \subseteq acl(Cb(c/B))$.

A 1-based theory is CM-trivial. E. Hrushovski's new strongly minimal theory is CM-trivial but not 1-based. (For all of the above see [12].) If T_c is the theory of a model of T with constants for some fixed finite sequence c, then T is CM-trivial iff T_c is CM-trivial, as mentioned by A. Pillay in [21]. Furthermore he mentioned in that paper that it is sufficient to replace $(CMT3)$ by the following:

$(CMT3)$**:** Let c be a tuple in the monster model $\mathbf{C}$ of T. Assume that $M \preceq N$ are elementary submodels of $\mathbf{C}$ with $acl(cM) \cap N = M$. Then $Cb(tp(c/M)) \subseteq acl(Cb(tp(c/N)))$.

Let G be any model of Σ. Let a_1, a_2, and a_3 be three elements in G that are linearly independent over $Z(G)$. Let Σ^* be the elementary theory of $(G; a_1, a_2, a_3)$ with constants for the three fixed elements. Note that Σ^* is independent from the choice of G and the elements a_1, a_2, a_3, because of axiom $(\Sigma3)$ and Lemma 7.5. Let $\overline{\Sigma^*}$ be the elementary theory of the vector space $\overline{G}$ with the fixed elements a_1/Z, a_2/Z, a_3/Z and the relation $R(x, y, z)$ defined by $[a_1/Z, x] = [y, z]$, where $[\ ,\ ]$ is the alternating bilinear map on $\overline{G} \times \overline{G}$ into $Z(G)$ given by the commutator. Let Γ be this interpretation of $\overline{\Sigma^*}$ in Σ^*.

Lemma 8.1. *There is an interpretation Λ of Σ^* in $\overline{\Sigma^*}$ such that every model $(G; a_1, a_2, a_3)$ of Σ^* is isomorphic to $\Lambda(\Gamma(G; a_1, a_2, a_3))$ and for every model M of $\overline{\Sigma^*}$ there is an isomorphism between M and $\Gamma(\Lambda(M))$ definable in M^{eq}.*

Proof. Let $(G; a_1, a_2, a_3)$ be a model of Σ^* and let B_G be the corresponding alternating bilinear map in $\mathcal{B}$ of $\overline{G} \times \overline{G}$ into $Z(G)$ with three fixed constants. By $(\Sigma2)$ and $(\Sigma3)$, $[a_1/Z, x]$ defines a vector space isomorphism of $\overline{G}/\langle a_1/Z \rangle$ onto $Z(G)$. Hence $R(x, y, z)$ gives an interpretation of B_G in $\Gamma(G)$. To prove the assertion we define an interpretation of G in B_G. We write the vector spaces $\overline{G}$ and $Z(G)$

additively as above. Starting with B_G we define in B_G a group G^+ isomorphic to G. G^+ consists of all tuples (a, x) with $a \in \overline{G}$ and $x \in Z(G)$. The multiplication is given by $(a, x) \circ (b, y) = (a + b, x + y + [a, b])$. It is easily seen that this defines a group. $(0, 0)$ is the unit and $(-a, -x)$ is the inverse element of (a, x). Furthermore $[(a, x), (b, y)] = (0, 2[a, b])$. G^+ fulfills $(\Sigma 1)$ and $(\Sigma 2)^w$. If $g_0, g_1, g_2, \ldots$ is a basis of $\overline{G}$, then the images of $(g_0, 0), (g_1, 0), (g_2, 0), \ldots$ in $\overline{G}^+$ form a basis of $\overline{G}^+$. A linear combination of commutators over $g_0, g_1, g_2, \ldots$ is 0 if and only if the corresponding linear combination of commutators over $(g_0, 0), (g_1, 0), (g_2, 0), \ldots$ is 0. By Lemma 3.1 G and G^+ are isomorphic, as desired. The definition of G^+ in $\Gamma(G)$ gives the desired interpretation Λ. $\qquad \square$

By Lemma 8.1 we have that $\overline{\Sigma^*}$ is the theory of the full structure definable on $\overline{G}$ in $(G; a_1, a_2, a_3)$.

Now we consider a model $\Gamma(G)$ of $\overline{\Sigma^*}$ living in a model G of Σ^* via the given interpretation. We do not mention the fixed three elements, but we will need them later. We get that $\overline{\Sigma^*}$ is strongly minimal. Furthermore an automorphism χ of $\Gamma(G)$ corresponds to an automorphism of the alternating bilinear map B_G living in G. By Lemma 3.1 we can extend χ to an automorphism of G. Hence Γ is an *interpretation without new information* in the sense of [3]. As shown in [4], forking for subsets of $\Gamma(G)$ does not depend on whether we consider them in $\overline{\Sigma^*}$ or in Σ^*.

Since we have fixed three linearly independent elements in every model M of $\overline{\Sigma^*}$, $acl(\emptyset)$ is infinite in M (in the real sort!). As mentioned by A. Pillay in [26], a strongly minimal theory with this property allows weak elimination of imaginaries. This notion was introduced by B. Poizat in [25]. We will use this property to prove CM-triviality of $\overline{\Sigma^*}$. First we will show that CM-triviality of $\overline{\Sigma^*}$ will imply CM-triviality of Σ. For this we will use the third form of the definition.

Lemma 8.2. *If $\overline{\Sigma^*}$ is CM-trivial, then Σ is CM-trivial.*

Proof. Assume that $\overline{\Sigma^*}$ is CM-trivial. We will show the CM-triviality of Σ^*. Then the *CM-triviality* of Σ follows, as mentioned above (see [21]). We use $(CMT3)$. Assume that c is a tuple in the monster model $\mathbf{C}$ of Σ^*. Let $G \preceq H$ be elementary submodels of $\mathbf{C}$ such that $acl(cG) \cap H = G$. We have to show that $Cb(tp(c/G)) \subseteq Cb(tp(c/H))$. Since in a model of Σ^* every tuple is interalgebraic with a tuple of elements that are linearly independent over $Z + \langle\{a_1, a_2, a_3\}\rangle$, we can assume that the elements of c are linearly independent over $Z + \langle\{a_1, a_2, a_3\}\rangle$. Let us write $\Gamma(c)$ for the image of c in the monster model $\mathbf{\Gamma(C)}$ of $\overline{\Sigma^*}$. Let $\Gamma(G)$ and $\Gamma(H)$ be the elementary submodels of $\mathbf{\Gamma(C)}$ that correspond to G and H respectively. Again $\Gamma(G) \preceq \Gamma(H)$, and in $\mathbf{\Gamma(C)}$ we have $acl(\Gamma(c)\Gamma(G)) \cap \Gamma(H) = \Gamma(G)$. $(CMT3)$ for $\overline{\Sigma^*}$ implies $Cb(tp(\Gamma(c)/\Gamma(G))) \subseteq acl(Cb(tp(\Gamma(c)/\Gamma(H))))$ By the weak elimination of imaginaries for $\overline{\Sigma^*}$ we have finite subspaces $A_0 \subseteq A_1$ of $\mathbf{\Gamma(C)}$ such that $A_0 \subseteq \Gamma(G)$ is interalgebraic with $Cb(tp(\Gamma(c)/\Gamma(G)))$ and $A_1 \subseteq \Gamma(H)$ is interalgebraic with $Cb(tp(\Gamma(c)/\Gamma(H)))$. Now we consider A_0 and A_1 as subspaces in the sort $\mathbf{C}/Z(C)$ in $\mathbf{C}$. We claim that A_0 is interalgebraic with $Cb(tp(c/G))$ and A_1 is interalgebraic with $Cb(tp(c/H))$. Then the assertion of the lemma follows. We consider the first case. The second works in the same way.

Let K be any elementary extension of G and let $tp(d/K)$ be the non-forking extension of $tp(c/G)$. We have to show that for all possible K and d above:

(i) Let Π be the set of all automorphisms of the monster model $\mathbf{C}$ that fixes A_0 pointwise and K setwise. Then $\{tp(\pi(d)/K) \ : \ \pi \in \Pi\}$ is finite.

(ii) Let Ξ be the set of all automorphisms ξ of the monster model $\mathbf{C}$ such that $tp(\xi(d)/K) = tp(d/K)$ and ξ fixes K setwise. Then $\{\xi(A_0) : \xi \in \Xi\}$ is finite.

First we prove (i). If we restrict all $\pi \in \Pi$ to $\mathbf{\Gamma(C)}$, then $\{tp(\pi(\Gamma(d))/\Gamma(K)) : \pi \in \Pi\}$ is finite since $Cb(tp(\Gamma(c)/\Gamma(G))) \subseteq acl(A_0)$. Assume $\pi_1, \pi_2 \in \Pi$ and $tp(\pi_1(\Gamma(d))/\Gamma(K)) = tp(\pi_2(\Gamma(d))/\Gamma(K))$. It remains to show that $tp(\pi_1(d)/K) = tp(\pi_2(d)/K)$. Let B be any finite subgroup of K with $(\Sigma 2)^w$ that contains a_1, a_2, and a_3. In $\mathbf{\Gamma(C)}$ we have an isomorphism η of

$$CSS(\langle\Gamma(\pi_1(d))\rangle + \Gamma(B)) \text{ onto } CSS(\langle\Gamma(\pi_2(d))\rangle + \Gamma(B))$$

that is the identity on

$$CSS(\langle\Gamma(\pi_1(d))\rangle + \Gamma(B)) \cap \Gamma(K) = CSS(\langle\Gamma(\pi_2(d))\rangle + \Gamma(B)) \cap \Gamma(K)$$

and that fulfills $\eta(\pi_1(d)) = \pi_2(d)$. By Lemma 7.5 we get

$$tp(\pi_1(d)/K) = tp(\pi_2(d)/K),$$

as desired.

(ii) follows immediately from the corresponding statement in $\mathbf{\Gamma(C)}$. $\qquad\square$

The following lemma is an essential part of the proof of the CM-triviality of $\overline{\Sigma^*}$.

Lemma 8.3. *Let A and B be finite $\mathcal{S}$-structures that satisfy $(\Sigma 3)$, where B is a minimal extension of A. Assume that $b_1, \dots, b_n$ is a linear basis of B over A, and that $N(B)$ is freely generated over $N(A)$ by relations $\Delta_1, \dots, \Delta_n$ of the following form:*

$$\Delta_i = \sum_{1 \leq j \leq n} a_j^i \wedge b_j + \Psi_i,$$

where $a_j^i \in A$ and $\Psi_i \in \wedge^2 A$. Finally, assume that it is impossible to present $N(B)$ over $N(A)$ as above without the use of nontrivial Ψ_i's. Then $(\Sigma 4)$ for A implies $(\Sigma 4)$ for B.

Note first that we can replace $b_1, \dots, b_n$ by any other basis of B over A. The basis of $N(B)$ over $N(A)$ we get with respect to the new basis of B over A has again the form described above and the new elements replacing the a_j^i's generate again the subspace $\langle\{a_j^i : 1 \leq i, j \leq n\}\rangle$ of A. Hence this subspace is uniquely determined by B.

Proof. Let $YX_1^+X_2^+X_2^- \dots X_h^+X_h^-$ be a reduced k-system in B. If it is completely in A, then we are done. Otherwise we choose a basis Y^aY^b of $\langle Y\rangle$ such that $\langle Y^a\rangle = \langle Y\rangle \cap A$. If there is some X_i^+ such that $X_i^+ \subseteq A$ modulo $\langle Y\rangle$, then w.l.o.g. we can assume that $i = 1$. Use (S3) and (S4). Using a well-tolerated automorphism we can assume that $X_1^+ = X^aX^b$, where Y^aX^a is a basis of $\langle YX_1^+\rangle \cap A$. For $2 \leq i \leq h$ we choose a basis V_iU_i of $\langle YX_1^+X_2^+X_2^- \dots X_i^+X_i^-\rangle$ over $\langle YX_1^+X_2^+X_2^- \dots X_{i-1}^+X_{i-1}^-\rangle$ such that $V_i \subseteq A$ and U_i is linearly independent over $\langle YX_1^+X_2^+X_2^- \dots X_{i-1}^+X_{i-1}^-\rangle + A$. To prove the next claim we use the special form of the free generators of $N(B)$ over $N(A)$.

(1) Assume $k > 0$. Then for $2 \leq i \leq h$ $V_i \neq \emptyset$.

Proof of (1). Assume $V_i = \emptyset$. We consider the condition (S1). Then $X_i^+X_i^-$ is part of a basis of B over $A + \langle YX_1^+X_2^+X_2^- \dots X_{i-1}^+X_{i-1}^-\rangle$. Since $k > 0$ there are $1 \leq s < t \leq k$ such that $\alpha_{st}^l(x_s^i \wedge x_t^i)$ occurs in Φ_l^i in (S1), where $\alpha_{st}^l \neq 0$. But

then it is impossible to get Φ_l^i as a linear combination of the Δ_j's above modulo $\wedge^2 A$.   □

(2) Assume $V_i \neq \emptyset$ for $2 \leq i \leq h$. Then $\langle Y X_1^+ \rangle \not\subseteq A$.

Proof of (2). We assume toward a contradiction that $\langle Y X_1^+ \rangle \subseteq A$. Since the reduced k-sequence under consideration is not contained in A, there exists a minimal i_0 such that $2 \leq i_0 \leq h$ and $U_{i_0} \neq \emptyset$. Since $\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i_0}^+ X_{i_0}^- \rangle$ is a minimal extension of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i_0-1}^+ X_{i_0-1}^- \rangle$, $V_{i_0} \neq \emptyset$, and $U_{i_0} \neq \emptyset$, we get

$$\dim(N(\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i_0}^+ X_{i_0}^- \rangle))$$
$$- \dim(N(\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i_0-1}^+ X_{i_0-1}^- V_{i_0} \rangle)) > |U_{i_0}|.$$

But then $\delta(A + \langle U_{i_0} \rangle) < \delta(A)$, a contradiction to the minimality of B over A.   □

The next assertion together with (1) implies the assertion of the lemma in the case $0 < k$.

(3) Assume that $V_i \neq \emptyset$ for $2 \leq i \leq h$. Then $h < |Y| + k + 2$, as desired.

Proof of (3). Assume $|Y| + k + 2 \leq h$. Let $n_i = |U_i|$. We show by induction on i (for $2 \leq i \leq h$) that:

($*_i$) There are m_i relations $\Theta_1, \ldots, \Theta_{m_i}$ in $N(Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^-)$ linearly independent over $N(A)$, where $m_i = \sum_{2 \leq j \leq i}(n_j + 1)$.

We prove the assertion for $i+1$ ($1 \leq i < h$). First assume $U_{i+1} \neq \emptyset$. Since $\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i+1}^+ X_{i+1}^- \rangle$ is a minimal extension of $\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle$ and $V_{i+1} \neq \emptyset$, there are $\Theta_{m_i+1}, \ldots, \Theta_{m_i+n_{i+1}+1} \in N(\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i+1}^+ X_{i+1}^- \rangle)$ which are linearly independent over $N(\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- V_{i+1} \rangle)$. Then

$$\Theta_{m_i+1}, \ldots, \Theta_{m_i+n_{i+1}+1}$$

are linearly independent over

$$N(\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- V_{i+1} \rangle) + \wedge^2 A,$$

since every nontrivial linear combination of them contains some $u \wedge x$ where $u \in U_{i+1}$. The assertion follows.

If $U_{i+1} = \emptyset$, then by (2) we get again $Y^b X^b \neq \emptyset$. We will show that we can find the desired Θ_{m_i+1} in

$$N(\langle Y X_1^+ X_2^+ X_2^- \ldots X_{i+1}^+ X_{i+1}^- \rangle) \setminus \wedge^2(\langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle) + \wedge^2 A$$

We work with the basis $Y^a Y^b X^a X^b U_2 V_2 \ldots U_i V_i V_{i+1}$. We consider Φ_j^{i+1} in (S1):

$$\sum_{1 \leq s < t \leq k} \alpha_{st}^j \left((x_s^{i+1} \wedge x_t^{i+1}) - (x_s^1 \wedge x_t^1) \right)$$

$$+ \sum_{1 \leq t \leq k} (y_t^j \wedge (x_t^{i+1} - x_t^1)) + \sum_{k < t \leq r} (y_t^j \wedge z_t^{i+1}).$$

By assumption $X^a X^b = X_1^+$. We can choose $V_{i+1} = \{v_1, \ldots, v_r\} \subseteq A$ in such a way that $x_t^{i+1} = v_t + c_t$ for $1 \leq t \leq k$ and $z_t^{i+1} = v_t + c_t$ for $k < t \leq r$, where $c_i \in \langle Y^b X^b U_2 \ldots U_i \rangle$. First we assume $Y^b \neq \emptyset$. Let y be an element of Y^b. We show that there is some j such that the representation of Φ_j^{i+1} with respect to the basis $Y^a Y^b X^a X^b U_2 V_2 \ldots U_i V_i V_{i+1}$ contains a basic commutator $y \wedge v_t$. Then $\Phi_j^{i+1} \notin \wedge^2 \langle Y X_1^+ X_2^+ X_2^- \ldots X_i^+ X_i^- \rangle + \wedge^2 A$ and the assertion follows. First we assume that

y occurs in the linear presentation of y_t^j for some $t > k$. Then $y \wedge v_t$ cannot vanish. Now we assume that, for all j, y only occurs in the presentation of y_t^j with respect to the basis $Y = Y^a Y^b$ if $t \leq k$. Then $y_t^j = \gamma_t^j y + w_t^j$ for $1 \leq j \leq r$ and $1 \leq t \leq k$, where $w_t^j \in \langle Y \setminus \{y\} \rangle$ and $y_t^j \in \langle Y \setminus \{y\} \rangle$ if $k < t$. For $1 \leq t \leq k$ we write $x_t^{i+1} = v_t + \epsilon_t y + d_t$, where $v_t \in V_{i+1}$ as above and $d_t \in \langle (Y^b \setminus \{y\}) X^b U_2 \dots U_i \rangle$. If we rewrite Φ_j^{i+1} with respect to the basis $Y X^a X^b U_2 V_2 \dots U_i V_i V_{i+1}$, then the coefficient of the basic commutator $y \wedge v_t$ is

$$\sum_{1 \leq s < t} \alpha_{st}^j \epsilon_t - \sum_{t < s \leq k} \alpha_{ts}^j \epsilon_t + \gamma_t^j.$$

If there is some j $(1 \leq j \leq r)$ and some t $(1 \leq t \leq k)$ such that this coefficient is not 0, then $\Phi_j^{i+1} \notin \wedge^2(\langle Y X_1^+ X_2^+ X_2^- \dots X_i^+ X_i^- \rangle) + \wedge^2 A$, as desired. If all these coefficients are 0, then we consider the following well-tolerated automorphism e: $e(x_t^i) = x_t^i - \epsilon_t y$ for $1 \leq t \leq k$, $e(z_t^i) = z_t^i$, and $e(a) = a$ for $a \in \langle Y \rangle$. If we rewrite Φ_j^i with respect to the basis $e(Y X_1^+ X_2^+ X_2^- \dots X_h^+ X_h^-)$, then

$$
\begin{aligned}
\Phi_j^i \;=\; & \sum_{1 \leq s < t \leq k} \alpha_{st}^j ((e(x_s^i) \wedge e(x_t^i)) - (e(x_s^1) \wedge e(x_t^1))) \\
& + \sum_{1 \leq t \leq k} (w_t^j \wedge (e(x_t^i) - e(x_t^1))) + \sum_{k < t \leq r} (y_t^j \wedge e(z_t^i)).
\end{aligned}
$$

Since $y_t^j \in \langle Y \setminus \{y\} \rangle$ for $k < t \leq h$ and $w_t^j \in \langle Y \setminus \{y\} \rangle$, we get (S1) with a smaller Y, a contradiction to (S4).

If $Y^b = \emptyset$, then $X^b \neq \emptyset$ by (2). By assumption $X^b \neq \emptyset$ implies that $X_{i+1}^+ \not\subseteq A$ modulo $\langle Y \rangle$. Hence there is some t_0 such that $1 \leq t_0 \leq k$ and $c_{t_0} \notin \langle Y \rangle$. We fix some u in $X^b U_2, \dots, U_i$ that occurs in c_{t_0}. Then we apply a well-tolerated automorphism to $\langle X_{i+1}^+ \rangle$ such that u occurs exactly in c_{t_0}. That means we subtract suitable $s x_{t_0}^{i+1}$ from each x_t^{i+1} with $t \neq t_0$ and $1 \leq t \leq k$. Then we choose some s and j with $\alpha_{t_0 s}^j \neq 0$ or $\alpha_{s t_0}^j \neq 0$ respectively. It follows that $u \wedge v_s$ will not vanish in Φ_j^{i+1}. Hence $\Phi_j^{i+1} \in N(\langle Y X_1^+ X_2^+ X_2^- \dots X_{i+1}^+ X_{i+1}^- \rangle) \setminus (\wedge^2(\langle Y X_1^+ X_2^+ X_2^- \dots X_i^+ X_i^- \rangle) + \wedge^2 A)$.

Now we compute $\delta(Y X_1^+ X_2^+ X_2^- \dots X_h^+ X_h^- + A) - \delta(A)$:

$$\dim(Y X_1^+ X_2^+ X_2^- \dots X_h^+ X_h^- + A) - \dim(A) = |Y^b| + |X^b| + \sum_{2 \leq j \leq h} n_j.$$

On the other hand by the induction above

$$\dim(N(\langle Y X_1^+ X_2^+ X_2^- \dots X_h^+ X_h^- \rangle + A)) - \dim(N(A)) \geq \left(\sum_{2 \leq j \leq h} n_j \right) + h - 1.$$

Hence

$$\delta(\langle Y X_1^+ X_2^+ X_2^- \dots X_h^+ X_h^- \rangle + A) - \delta(A) \leq |Y^b| + |X^b| - h + 1.$$

If $h \geq |Y| + k + 2$, then this number is negative, a contradiction to the selfsufficiency of A. $\qquad \square$

Now (1) and (3) imply the assertion for the case $0 < k$. We assume $k = 0$. If for all $2 \leq j \leq h$ we have $V_j \neq \emptyset$, then by (3) we obtain again the assertion. Hence we can assume w.l.o.g. that $V_2 = \emptyset$. Use (S2). We show that this is impossible. First we have to prove:

(4) If $k = 0$ and $V_2 = \emptyset$, then $Y \subseteq A$.

$V_2 = \emptyset$ implies that X_2^- is linearly independent over A. By (S1) the $\Phi_j^2 = \sum_{1 \le t \le r} (y_t^j \wedge z_t^2)$ for $1 \le j \le r$ generate $N(\langle YX_2^- \rangle)$ over $N(\langle Y \rangle)$ freely. If some y_t^j $(1 \le t \le r)$ is not in A, then it is impossible to write Φ_j^2 as a linear combination of $\Delta_1, \dots, \Delta_n$ modulo $\wedge^2 A$. Hence (4) is shown. $\qquad\square$

Now (4) implies that $\langle YX_2^- \rangle$ is a minimal extension of $\langle Y \rangle \subseteq A$. Then $\Phi_1^2, \dots, \Phi_r^2$ are linearly independent over $\wedge^2 A$. We have $\langle AX_2^- \rangle = B$ and $r = n$. But in the assumptions of the lemma we have excluded such a basis $\Phi_1^2, \dots, \Phi_n^2$ of $N(B)$ over $N(A)$ that only uses basic commutators $a_i \wedge b_i$ where $a_i \in A$ and $b_1, \dots, b_n$ is a basis of B over A. Hence we have the desired contradiction. $\qquad\square$

Lemma 8.4. $\overline{\Sigma^*}$ *is CM-trivial.*

Proof. Let M be a saturated model of $\overline{\Sigma^*}$. Then Lemma 8.1 gives a model G of Σ^* defined in M. G is saturated, and M is the induced structure on $\overline{G}$. We also use M to denote the corresponding $\mathcal{S}$-structure. The following proof uses the fact that M is strongly minimal, that $acl = cl$, and that independence in the sense of forking is geometrical independence.

Assume B_1, B_2, A, and E are algebraically closed sets in M^{eq}. Let $\overline{B_i}$ be $acl(B_i E)$. Furthermore assume that the following conditions are fulfilled:

  i) $acl(B_1 + B_2) \cap \overline{B_i} = B_i$.
  ii) $B_i \cap E = A$.
  iii) B_1 and B_2 are independent over E.

We have to show that B_1 and B_2 are independent over A.

Since $\overline{\Sigma^*}$ has weak elimination of imaginaries, every algebraically closed set is the algebraic closure of an algebraically closed set in the real sort. Hence we can work in the model M instead of M^{eq}. By iii)

$$\overline{B_1} + \overline{B_2} = \overline{B_1} \star_E \overline{B_2}$$

and $\overline{B_1} + \overline{B_2}$ is selfsufficient in M.

Since $\overline{B_1} \cap \overline{B_2} = E$ we obtain $B_1 \cap B_2 \subseteq E$. Hence $B_i \cap E = A$ implies $B_1 \cap B_2 = A$. It is sufficient to show that $B_1 + B_2$ is selfsufficient and

$$B_1 + B_2 = B_1 \star_A B_2.$$

W.l.o.g. we assume that all sets considered have finite geometrical dimension.

(5) $B_1 + B_2$ is selfsufficient .

Indeed, we know that $(\overline{B_1} + \overline{B_2}) \cap acl(B_1 + B_2)$ is selfsufficient. Hence we can work in this selfsufficient subspace. If $B_1 + B_2$ was not selfsufficient, then there would exist an extension D of $B_1 + B_2$ such that

  the linear dimension of D over $(B_1 + B_2)$ is finite,

  $D \subseteq acl(B_1 + B_2) \cap (\overline{B_1} + \overline{B_2})$, and

  $\delta(D) < \delta(B_1 + B_2)$.

Since $acl(B_1 + B_2) \cap \overline{B_i} = B_i$, D has a linear basis $\{u_1 + v_1, \dots, u_n + v_n\}$ over $B_1 + B_2$, where $u_1, \dots, u_n \in \overline{B_1}$ are linearly independent over $B_1 + E$ and $v_1, \dots, v_n \in \overline{B_2}$ are linearly independent over $B_2 + E$ (Lemma 4.2).

Since $\delta(D) < \delta(B_1 + B_2)$, there are $m > n$ many relation $\Phi_1, \dots, \Phi_m$ in $N(D)$ that are linearly independent over $\wedge^2(B_1 + B_2)$. By Lemma 4.3 there are a_i^j,

$\Psi_j^1 \in \wedge^2 B_1$, $\Psi_j^2 \in \wedge^2 B_2$, and $I_j \in \wedge^2 E$ for $1 \leq j \leq m$, $1 \leq i \leq n$ such that

$$\Phi_j = \sum_{1 \leq i \leq n} (a_i^j \wedge (u_i + v_i)) + \Psi_j^1 + \Psi_j^2,$$

$$\Phi_j^1 = \sum_{1 \leq i \leq n} (a_i^j \wedge u_i) + \Psi_j^1 + I_j \in N(\overline{B_1}),$$

$$\Phi_j^2 = \sum_{1 \leq i \leq n} (a_i^j \wedge v_i) + \Psi_j^2 - I_j \in N(\overline{B_2}).$$

Note that it follows that $A \neq \emptyset$. The linear independence of $\Phi_1, \dots, \Phi_m$ over $\wedge^2(B_1+B_2)$ implies that the $(\sum_{1 \leq i \leq n} a_i^j \wedge u_i)$ for $1 \leq j \leq m$ are linearly independent over $\wedge^2(B_1 + E)$ and the $(\sum_{1 \leq i \leq n} a_i^j \wedge v_i)$ for $1 \leq j \leq m$ are linearly independent over $\wedge^2(B_2 + E)$. Hence $\Phi_1^1, \dots, \Phi_m^1$ are linearly independent over $\wedge^2(B_1 + E)$ and $\Phi_1^2, \dots, \Phi_m^2$ are linearly independent over $\wedge^2(B_2 + E)$. W.l.o.g. we can assume that there is some $m_0 \leq m$ such that $I_j = 0$ for $1 \leq j < m_0$ and $I_{m_0}, \dots, I_m$ are linearly independent over $\wedge^2 A$. Then $m_0 \leq n \leq m$, since otherwise $\delta(B_1 + \langle u_1, \dots, u_n \rangle) \leq \delta(B_1)$ would imply $u_1, \dots, u_n \in B_1$, a contradiction. Assume that D is chosen of minimal size over $B_1 + B_2$. That means n is minimal. Let $B_1' \subseteq B_1$, $E' \subseteq E$, and $A' \subseteq A$ be finite selfsufficient subspaces such that $\delta(B_1') = d(B_1)$, $\delta(E') = d(E)$, $\delta(A') = d(A)$, $I_j \in \wedge^2 E'$, $\Psi_j^1 \in \wedge^2 B_1'$ (for $1 \leq j \leq m$, $1 \leq i \leq n$), and $E' \cap A = B_1' \cap A = A' = E' \cap B_1'$.

Let $(E', N(E'))$ be the $\mathcal{S}$-structure corresponding to E'. We define a finite $\mathcal{S}$-structure C that extends E'. We prove that C is an algebraic extension of E' that satisfies ($\Sigma 3$) and ($\Sigma 4$). C is freely generated by $c_1, \dots, c_n$ over E', and $N(C)$ is freely generated over $N(E')$ by $\Delta_1, \dots, \Delta_n$, where $\Delta_j = (\sum_{1 \leq i \leq n} a_i^j \wedge c_i) + I_j$ is obtained from Φ_j^1 ($1 \leq j \leq n$) above.

(6) $\delta(C) = \delta(E')$ and C satisfies ($\Sigma 3$).

We prove (6). The linear independence of $\Phi_1^1, \dots, \Phi_n^1$ over $\wedge^2(E + B_1)$ implies that $\Delta_1, \dots, \Delta_n$ are linearly independent over $\wedge^2 E'$. Hence $\delta(C) = \delta(E')$. For the proof of (6) we show for every subspace K of C:

(7) $\delta(K) \geq \delta(K \cap E')$.

If $\dim(K) - \dim(K \cap E') = n$, then (7) is clear. Assume $\delta(K) < \delta(K \cap E')$. We show a contradiction. Let K be freely generated by $d_1, \dots, d_l$ over $K \cap E'$. Then $l < n$ and there are $l + 1$ relations $\Theta_1, \dots, \Theta_{l+1}$ in $N(K)$ linearly independent over $\wedge^2(K \cap E')$. Note that every Θ_j is a linear combination of $\Delta_1, \dots, \Delta_n$ modulo $N(E')$. Every d_i is a linear combination d_i^+ of $c_1, \dots, c_n$ modulo E'. These d_i^+ are linearly independent over E'. If we consider the corresponding linear combinations w_i^+ of $u_1 + v_1, \dots, u_n + v_n$, then

$$\delta(B_1 + B_2 + \langle w_1^+, \dots, w_l^+ \rangle) < \delta(B_1 + B_2),$$

a contradiction to the minimality of n. This proves (7). $\qquad \square$

From (7) we get $\delta(K) \geq 3$, if $\dim(K \cap E') \geq 3$, because ($\Sigma 3$) is true for E'. Let us assume that $\dim(K \cap E') < 3$. Then

$$\delta(K \cap E') = \dim(K \cap E') \quad \text{and} \quad \dim(K \cap A') < 3.$$

We discuss the three possible cases for $\dim(K \cap A')$.

If $\dim(K \cap A') = 0$, then no nontrivial linear combination of $\Delta_1, \ldots, \Delta_n$ is in $N(K)$ modulo $N(E')$, since the $(\sum_{1 \leq i \leq n} a_i^j \wedge c_i)$'s are linearly independent. Then $\delta(K) = \delta(K \cap E') + \dim(K) - \dim(K \cap E') = \dim(K)$, as desired.

If $\dim(K \cap A') = 1$, then either $N(K) = N(K \cap E') = 0$ or there is a relation of the form $\Delta = a \wedge d + I$ in $N(K)$, where $\langle a \rangle = K \cap A'$, $I \in \wedge^2(E' \cap K)$ and $d = c$ modulo E', where c is a nontrivial linear combination of $c_1, \ldots, c_n$. Δ has this form because w.l.o.g. it is a linear combination of the Δ_i's . We can rewrite Δ as $a \wedge c + I'$, where $I' \in \wedge^2 E'$. We find a linear combination $\Psi \in N(E + B_1 + \langle \{u_1, \ldots, u_n\} \rangle)$ of $\Phi_1^1, \ldots, \Phi_n^1$ corresponding to Δ. That means that in the linear combination Δ we replace every Δ_i by the corresponding Φ_i^1. We get that Ψ is of the form $a \wedge u + \Psi^1 + I'$, where $\Psi^1 \in \wedge^2 B_1$, and u is a nontrivial combination of $u_1, \ldots, u_n$. Since E is algebraically closed, there is some $e \in E$ with $a \wedge e = I'$ (use $(\Sigma 2)$). Since B_1 is algebraically closed, there is some $b \in B_1$ with $a \wedge b = \Psi^1$. Hence $a \wedge (u + e + b) \in N(E + B_1)$. By $(\Sigma 3)$ for M we obtain $u + e + b = ra$. Then $u = ra - e - b \in E + B_1$. This is a contradiction, since $u_1, \ldots, u_n$ are linearly independent over $E + B_1$.

There remains the case $\dim(K \cap A') = 2$. Then $K \cap A' = K \cap E'$. If $\dim(K) = 2$, then $K = K \cap A'$ and $\delta(K) = \dim(K) = 2$, as desired. Otherwise $\dim(K) = r + 2 > 2$. We suppose $\delta(K) < 3$ and derive a contradiction. By (7) $\delta(K) = \delta(K \cap A') = 2$. Hence there are r linear combinations $\Theta_1, \ldots, \Theta_r$ in $N(K)$ linearly independent over $\wedge^2(K \cap A')$. There are linear combinations $b_1, \ldots, b_r$ of $c_1, \ldots, c_n$ that are linearly independent and elements $e_1, \ldots, e_r \in E'$ such that

$$K = (K \cap A') + \langle b_1 + e_1, \ldots, b_r + e_r \rangle.$$

Since we can replace $c_1, \ldots, c_n$ by any other basis of C over E', we can assume w.l.o.g. that $c_i = b_i$ and $e_i = 0$ for $1 \leq i \leq r$. Since every Θ_i has to be a nontrivial linear combination of the Δ_j modulo $N(E')$ we can assume w.l.o.g that $\Theta_1 = \Delta_1, \ldots, \Theta_r = \Delta_r$. $\Delta_i \in N(K)$ implies $I_i \in \wedge^2(K \cap A')$. Hence $\Phi_1^1, \ldots, \Phi_r^1 \in N(B_1 + \langle u_1, \ldots, u_r \rangle)$ are linearly independent over $N(B_1)$. Hence $u_1, \ldots, u_n \in B_1$, a contradiction. (6) is shown. $\square$

If C is not a minimal extension of E', then there are $E' = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_s = C$ such that C_{i+1} is a minimal extension of C_i for $0 \leq i < s$. Every C_{i+1} is freely generated by some $d_1, \ldots, d_l$ over C_i, and $N(C_{i+1})$ is freely generated over $N(C_i)$ by l relations of the form $(\sum_{1 \leq i \leq l} a_i^j \wedge d_i) + J_j$, where $a_i^j \in A'$ and $J_j \in \wedge^2 C_i$. These relations are linear combinations of $\Delta_1, \ldots, \Delta_n$ modulo $N(E')$. We cannot choose $d_1, \ldots, d_l$ in such a way that $J_1 = \ldots = J_l = 0$. Otherwise we could choose the corresponding $u_1, \ldots, u_l$ in B_1, a contradiction. Hence we can apply Lemma 8.3 to every C_{i+1} over C_i. We have $(\Sigma 4)$ for all C_i:

(8) C satisfies $(\Sigma 4)$.

Now we can apply axiom $(\Sigma 5)$ successively to every C_{i+1} over C_i. Since M was chosen saturated, there is a selfsufficient image of C in M. We denote it again by C. Since E was algebraically closed, $C \subseteq E$. Hence we have $c_1, \ldots, c_n$ in E such that for $1 \leq j \leq n$

$$\left(\sum_{1 \leq i \leq n} a_i^j \wedge c_i \right) + I_j \in N(E).$$

It follows that

$$(\sum_{1 \leq i \leq n} a_i^j \wedge (u_i - c_i)) + \Psi_j^1 \in N(\overline{B_1}).$$

Since these relations are linearly independent over $N(B_1')$, $a_i^j \in A'$, and $\Psi_j^1 \in \wedge^2 B_1'$, we obtain that $u_i - c_i$ are in the algebraic closure of B_1', i.e., in B_1. We have shown that $u_i = c_i + (u_i - c_i) \in E + B_1$, a contradiction. Hence $B_1 + B_2$ is selfsufficient, as desired. □

It remains to show that $B_1 + B_2 = B_1 \star_A B_2$. This follows from:

(9) $\delta(B_1 + B_2) = \delta(B_1) + \delta(B_2) - \delta(A)$.

By Lemma 3.4 we have $\delta(B_1 + B_2) \leq \delta(B_1) + \delta(B_2) - \delta(A)$. To show equality we consider some Φ in $N(B_1 + B_2)$. Since we work in $\overline{B_1} \star_E \overline{B_2}$ we obtain by Lemma 4.3 $\Phi = \Phi^1 + I + \Phi^2 - I$, where $\Phi^1 + I \in N(\overline{B_1})$, $\Phi^2 - I \in N(\overline{B_2})$, $\Phi^1 \in \wedge^2 B_1$, $\Phi^2 \in \wedge^2 B_2$, and $I \in \wedge^2 E$. Let a be one of the fixed constants in M. By ($\Sigma 2$) there is some $e \in E$ such that $a \wedge e + I \in N(E)$. Then $\Phi^1 - a \wedge e \in N(\overline{B_1})$. Since $a \in B_1$, $\Phi^1 \in \wedge^2 B_1$, and B_1 is algebraically closed we obtain $e \in B_1$. On the other hand $\Phi^2 + a \wedge e \in N(\overline{B_2})$. Similarly as above we obtain $e \in B_2$. Hence $e \in B_1 \cap B_2 = A$. Then $\Phi = \Phi^1 - a \wedge e + \Phi^2 + a \wedge e$, where $\Phi^1 - a \wedge e \in N(B_1)$ and $\Phi^2 + a \wedge e \in N(B_2)$, as desired. CM-triviality of $\overline{\Sigma^*}$ is shown. □

Lemma 8.2 and Lemma 8.4 imply

Theorem 8.5. Σ *is CM-trivial.*

CM-triviality implies the non-interpretability of a field and of a simple group in the case of finite Morley rank, as proved by A. Pillay [21]. Hence from Theorem 8.5 we conclude:

Theorem 8.6. *No field is interpretable in* Σ.

Together with Theorem 2.1 we obtain:

Corollary 8.7. *Every group interpretable in* Σ *is a direct sum of a group that is nilpotent by finite and of bounded exponent, and an abelian group.*

Furthermore we have proved in [6]:

Theorem 8.8. *Every group interpretable in* Σ *is nilpotent by finite and of bounded exponent.*

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